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Orbit Determination and Analysis by the Minimum Variance Method

by

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ERRATA

Page 11, Eo.(6) read $\tan \phi = \frac{h + f_E C}{h + C} \tan \Phi$

Page 64, Eq.(163) for $X_T = \ldots$ read $X_T'' = \ldots$

Page 65, Ec.(164) for $Y_T = \ldots$ read $Y_T'' = \ldots$

Page 66, Eq.(165) for $Z_T = \ldots$ read $Z_T'' = \ldots$

For clarification, Eqs.(166a,b,c) on pages 66-67 should read

$$X_G = -\frac{\mu x}{r^3} + X_Z + X_T'' \frac{\partial x}{\partial x}'' + Y_T'' \frac{\partial y}{\partial x}'' + Z_T'' \frac{\partial z}{\partial x}'' \qquad (166a)$$

$$Y_G = -\frac{\mu y}{r^3} + Y_Z + X_T'' \frac{\partial x}{\partial y}'' + Y_T'' \frac{\partial y}{\partial y}'' + Z_T'' \frac{\partial z}{\partial y}'' \qquad (166b)$$

$$Z_G = -\frac{\mu z}{r^3} + Z_Z + X_T'' \frac{\partial x}{\partial z}'' + Y_T'' \frac{\partial y}{\partial z}'' + Z_T'' \frac{\partial z}{\partial z}'' \qquad (166c)$$

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DATA ANALYSIS BRANCH (CRMXA)
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

ABSTRACT

The aspects of accurate determination of Earth satellite orbits by the Minimum Variance Method are presented. In addition, techniques for the determination of the associated physical constants, such as the coefficients in the Earth's gravitational potential, exospheric temperature, etc., are developed. A method for determination of the state transition matrix is presented. Also included are a review of the time systems employed in satellite orbit determination and a short discussion of the types of observations.

The mathematical model of the dynamical system includes nine zonal harmonics and up to the fourth order tesseral harmonics of the Earth's gravitational potential. Atmospheric drag effects are included on the assumption that the atmosphere rotates with the angular velocity of the Earth. First order solar and lunar gravitational attractions and solar radiation pressure are also treated. The satellite orbits are integrated in a reference system which considers the precession and nutation of the Earth. Rectangular coordinate systems are used throughout the development.

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FOREWORD

This report presents the analytical foundations for several computer programs now under development. The research is being performed for the Data Analysis Branch (CRMXA), Technical Services Division at AFCRL, USAF, L. G. Hanscom Field, Bedford, Massachusetts. The digital computer programming in this research effort is being done by Bruce Clemenz and Jacques Fein. The contractor's report number is ER 13950.

CONTENTS

		Page
	Abstract	iii
	Foreword	v
I.	Introduction	1
II.	Time Systems	3
	A. Measurement of Time	3
	B. Ephemeris Time	3
	C. Atomic Time	3
	D. Sidereal Time	4
	E. Universal Time	4
	F. Radio Time Signals	6
III.	Space Reference Systems	9
	A. System Requirements	9
	B. Basic Reference System	9
	C. Terrestrial Reference System	9
	D. Station Position in the Basic System	11
	E. Precession and Nutation	12
	F. Observation Reference Systems	15
IV.	Satellite Observations	19
	A. Classification of Observations	19
	B. Electronic Observations	19
	C. Optical Observations	20
	D. Observation Corrections	20

CONTENTS (continued)

		Page
V.	Application of the Minimum Variance Method	23
	A. Formulation of the Problem	23
	B. Some Practical Aspects	24
	C. Linear Concepts	24
	D. Filtering Equations	27
	E. Rejection of Observations	30
	F. Determination of Constants, Type 1	31
	G. Determination of Constants, Type 2	34
VI.	State Transition Matrix	39
	A. Formulation of the Problem	39
	B. Elliptic Orbits	39
	C. Circular Orbits	46
	D. Parabolic Orbits	47
	E. Hyperbolic Orbits	47
VII.	Equations of Observations	49
	A. Requirements for Analysis	49
	B. Analytic Expressions	49
VIII.	Equations of Motion	57
	A. Aspects of the Mathematical Model	57
	B. Earth's Gravitational Field	57
	C. Atmospheric Drag	67
	D. Sun's Gravitational Attraction	68

CONTENTS (continued)

		Page
	E. Moon's Gravitational Attraction	70
	F. Solar Radiation Pressure	71
	G. Integration of the Equations of Motion	72
IX.	References	75
X.	Appendix	77

I. INTRODUCTION

The orbit of a satellite, in a given reference system, is determined by six parameters, called the orbital elements. These elements can be determined from a sufficient number of other parameters obtained as observations. However, the problem is usually either under- or over-determined. In addition, the observations contain errors. For these reasons, normal algebraic methods cannot be employed, and the problem enters the realm of statistics and probability. Satellite observations are used not only to determine the orbit, but also to estimate the various physical and other constants of the dynamical and the observation systems. Precise orbit determination and analysis by a statistical filtering technique known as the Minimum Variance Method is the subject of this report.

The motion of a satellite in an inverse-square central force field is represented by conic sections. This motion would be realized if the sole force acting on the satellite were due to a point mass or a homogeneous sphere. In nature, however, the orbit of a satellite is perturbed by a variety of forces, e.g., harmonics in the gravitational potential, atmospheric drag, attractions of other celestial bodies, etc. Consequently, the dynamical system is much more complicated.

As mentioned before, the orbit of a satellite is determined by the six orbital elements. These elements can represent an orbit in a simple inverse-square central force field as well as in the actual, complex dynamical system. Any set of independent parameters which uniquely describe an orbit can be used as orbital elements. For example, the six classical elements are: a, the semi-major axis; e, the eccentricity; Ω, the right ascension of the ascending node; ω, the argument of pericenter; i, the inclination; and M, the mean anomaly. The orbits in the present report are considered in a rectangular coordinate system. Consequently, the orbital elements in this system are most conveniently represented by the six components of the position and velocity vectors, which are the standard elements in this analysis. The certain numerical difficulties experienced with these elements in the Least Squares application are not encountered employing the Minimum Variance technique.

The standard reference epoch for the celestial equations in this report is 1950 January 1, 0^h UT, except as noted.

The present report deals specifically with the orbit determination of close Earth satellites, including parabolic and hyperbolic orbits. The same principle, however, applies to satellites of other celestial bodies.

II. TIME SYSTEMS

A. MEASUREMENT OF TIME

The accurate measurement of time has been the concern of a rather specialized field. However, with the advent of the artificial satellite, a new area has been introduced which is directly concerned with the precise aspects of time and its measurement. In addition, there have been many developments in the time systems during the last few years. Therefore, a review of this subject is necessary. A thorough treatment can be found in Refs. 1, 2 and 3. The present discussion will deal only with the existing time systems and their application to satellite orbit determination and analysis.

In the measurement of time, two important factors are involved. One is the epoch or reference from which time is measured. The other is the rate at which time is measured. A system used to measure time, therefore, must be some observable physical phenomenon that has a well-defined epoch and whose rate of change is as invariant as possible. There are seven fundamental time systems in existence: ephemeris time (ET), atomic time (A.1), true sidereal time, mean sidereal time, UT0, UT1 and UT2. The last three are subdivisions of universal time. In addition, there are time systems emitted as signals on several radio frequencies throughout the world: WWV, NBA (United States), GBR (United Kingdom), etc.

B. EPHEMERIS TIME

Ephemeris time is the uniform time of dynamical astronomy. Theoretically, it is the independent time argument of the ephemerides of the Sun, Moon and the planets. In practice, it is determined from the orbital motion of the Moon about the Earth. This involves the solution of the equations of motion of the Moon. Thus, the uniformity of the time argument will be dependent on the accuracy of the representation of the system in which the Moon moves. Since it also involves observations of the Moon, additional errors are introduced. The errors are not significant, and the accuracy of the determined ET is believed to be within a couple of seconds in a century.

Ephemeris time would be the proper time argument in the equations of motion of an Earth satellite. However, its determination requires several years of observations and it is not practical for observing events of relatively short time intervals apart without an intermediary.

C. ATOMIC TIME

In 1955, a precise, cesium-133 atomic resonator was introduced in Great Britain. Since then, nine laboratories in five countries have been operating

cesium resonators in a coordinated effort. It is estimated that the accuracy of the cesium resonators is within 0.001 second in three years. Because of the exceptional accuracy of this system (designated A.1), the 12th General Conference on Weights and Measures, in December 1964, changed the basis of the definition of the second from ephemeris to atomic time scale. Thus, a uniform time system has been made available which provides exceptional stability and convenience.

The A.1 system is defined as follows (Ref. 2):

(1) A clock which keeps A.1 time advances one second in the interval required for 9,192,631,770 oscillations of cesium at zero field.

(2) At $0^h\,0^m\,0^s$ UT2 on 1 January 1958, the value of A.1 was $0^h\,0^m\,0^s$.

Because the atomic time system is uniform for all practical purposes and readily available, it is the most suitable for use as the independent time argument in the satellite equations of motion.

The rapid progress in atomic timekeeping is indicated by the announcement, even as this report was being written, that the U. S. Naval Research Laboratory has installed twin atomic hydrogen masers to continuously reset the master clock at the U. S. Naval Observatory. The hydrogen masers will keep the clock accurate to within one second in 300,000 years.

D. SIDEREAL TIME

Sidereal time is directly related to the rotation of the Earth. It is defined as the hour angle of the vernal equinox. Thus, except for small motions of the equinox itself, sidereal time is a direct measure of the diurnal rotation of the Earth. Sidereal time measured with respect to the true equinox is true or apparent sidereal time. If measured with respect to the mean equinox of date, it is called mean sidereal time. If the meridian is that of Greenwich, it is called Greenwich mean sidereal time. For any other meridian, it is the local sidereal time. Because sidereal time reflects the variable rotation of Earth, there is no direct relationship between sidereal time and ephemeris time. In orbit determination, the computation of sidereal time is required in order to determine the position of the observing station.

E. UNIVERSAL TIME

Universal time is based on the diurnal motion of the Sun and is the basis of all civil timekeeping. It involves both the rotation of Earth and the motion of Earth in its orbit about the Sun. Universal time and sidereal time are equivalent systems and are directly related to each other by means of a

numerical formula. Universal time is defined as the Greenwich hour angle of a point on the equator whose right ascension, measured from the mean equinox of date, is

$$R_u = 18^h\,38^m\,45^s.836 + 8640184^s.542\,T_u + 0^s.0929\,T_u^2 \qquad (1)$$

Time, T_u, is given in Julian centuries of 36,525 days of universal time elapsed since the epoch 1900 January 0, 12^h UT. The practical determination of universal time is made through the intermediary of sidereal time by observing the diurnal motions of the stars. The relation between Greenwich mean sidereal time and UT is given by the equation:

$$\text{Greenwich mean sidereal time} = UT + R_u + 12^h \qquad (2)$$

The sidereal times are computed in advance for successive dates of 0^h UT from the above equations and published in the American Ephemeris and Nautical Almanac. Local mean sidereal time at any particular instant is obtained from observations of the transit of stars of known positions. Greenwich mean sidereal time is obtained by adding the longitude west of Greenwich. Then the corresponding universal time is obtained by taking the difference in sidereal times at the instant of the observation and the computed value at 0^h UT and converting it to UT by the known relationship. The universal time thus obtained is designated UT0. Due to the movement of the Earth's axis of rotation, known as polar motion, and variation in the rate of rotation of the Earth, UT0 contains irregularities. Although the polar motion is very small, it affects the time measurements, which can be now performed with great accuracy. UT0 corrected for the polar motion to a mean Greenwich meridian is designated UT1.

The variations in the rate of rotation of the Earth are due to many causes. Some of them are negligible; some of them are quite large but unpredictable. The seasonal variation is a periodic variation of a maximum amplitude of approximately 0.03 second. It is a quite stable variation and can be predicted with good accuracy. UT1 corrected for the seasonal variations is designated UT2. Corrections for the years 1956 to 1962 have been computed from the formula (Ref. 1):

$$\begin{aligned}\Delta T = {} & +0^s.022 \sin 2\pi\tau - 0^s.017 \cos 2\pi\tau \\ & - 0^s.007 \sin 4\pi\tau + 0^s.006 \cos 4\pi\tau\end{aligned} \qquad (3)$$

Since 1962, the following formula has been used (Ref. 4):

$$\begin{aligned}\Delta T = {} & +0^s.022 \sin 2\pi\tau - 0^s.012 \cos 2\pi\tau \\ & - 0^s.006 \sin 4\pi\tau + 0^s.007 \cos 4\pi\tau\end{aligned} \qquad (4)$$

where τ is the fraction of the year and is zero on January 1.

It is important to note that universal time is not a uniform time and therefore cannot be properly used as the independent time argument in the equations of motion. However, UT1 is of particular significance in accurate satellite orbit determination. Since UT1 is obtained by observing the rotational position of Earth with respect to stars, the reverse process is applied to determine the position of Earth from UT1.

As noted before, only the ratio of sidereal time and universal time can be expressed by a numerical formula. There are no rigorous analytical relationships for the other systems. The difference between ephemeris time and universal time presently is about 35 seconds. The difference between A.1 and UT2 was 2.3385 seconds on January 0, 1963, increasing by about 0.5 second per year.

F. RADIO TIME SIGNALS

Neither of the time systems discussed above is directly available to the user. Instead, time signals are emitted by special radio stations (WWV, WWVH and NBA in the United States) in accordance with international agreements. Prior to 1959, station WWV emitted time signals at a constant frequency while making phase adjustments of 0.02 second when necessary to keep the signals close to UT2. The signals now are emitted with a frequency maintained constant each year but offset with reference to the atomic time standard. The time pulses are kept within approximately 0.1 second of UT2. In addition, phase adjustments of the pulses can be made if necessary. There is no analytical relationship between WWV and UT2. For this reason, the differences are given in periodical Time Service Bulletins issued by the U. S. Naval Observatory.

The methods used in satellite observation and timing are continuously improving. It is estimated (Ref. 5) that the position of a satellite can be measured with an accuracy of 20 meters. For practical reasons, the timing of a satellite observation is done by the clock of a station which observes the satellite. Although the received WWV signals will be in error with respect to the emitted signals because of uncertainty in propagation, it is estimated (Ref. 5) that a worldwide tracking system can be synchronized to an accuracy of about 0.001 second. To fully utilize the accuracy available in precision orbit determination, the proper corrections should be applied to the observation times to arrive at the correct time systems required in the analysis. These corrections, however, are not immediately available and in such cases they must be either extrapolated or the time recorded by the station clock used as an approximation. During the period in which there are no changes either in frequency or phase of the WWV signals, the recorded time will be essentially uniform, if the station clocks are well synchronized with the WWV. Therefore, this time could be used as the independent time argument in the satellite equations of motion. An error is introduced by substituting this time for UT1 to compute the station position. For example, the position of a station at 30° latitude can be in error by as much as 50 meters. The numerical values for the masses of

Earth and other celestial bodies are not sufficiently well known at the present time to be affected by the small differences in the time systems, but this may change in the future.

The following procedure can be used to obtain systems A.1 and UT1 required for accurate orbit analysis:

(1) Time recorded by the station clock corrected to obtain WWV emitted signal

(2) WWV emitted signal corrected to obtain A.1

(3) WWV or A.1 corrected to obtain UT2

(4) UT2 corrected to obtain UT1.

Correction (1) includes corrections, if any, to the recorded time to obtain WWV received. This time is then corrected for propagation effects to obtain WWV emitted. Corrections (2), (3) and (4) are published in the Time Service Bulletins. Correction (4) can be applied by means of Eqs. (3) and (4).

III. SPACE REFERENCE SYSTEMS

A. SYSTEM REQUIREMENTS

There are several basic reference systems used in orbit determination. The satellite motion itself must be ultimately considered in an inertial system. The position of the observer must be referenced to a terrestrial system, and the observations are obtained in either a geodetic or a celestial system. In the final analysis, the relationship between these systems must be introduced.

B. BASIC REFERENCE SYSTEM

The satellite orbit could be referenced to a fixed geocentric celestial system. However, such a system would not be very convenient for determining the forces acting on the satellite because of the precession and nutation of the Earth. The expressions for the forces could be much simplified if the motion is considered in a moving axis system defined by the true equator and equinox. In such a system, however, a supplementary or Coriolis acceleration is introduced. Since the rate of precession and nutation is comparatively small, we will utilize a system which is considered inertial for a short period of time and coincides with a mean position of the true equator and equinox during this interval. The satellite orbit thus is considered in a system which moves stepwise with the rate of precession and nutation. The length of the interval can be made as short as is necessary and practical. The only errors introduced will be due to the small variation of the gravitational field caused by precession and nutation during this interval. However, for all practical purposes, these errors will be negligible. In addition to giving minimum errors, this system is ideally suited to the point-to-point technique of the Minimum Variance Method and the integration method employed.

C. TERRESTRIAL REFERENCE SYSTEM

A terrestrial reference system will be defined as a rectangular right-hand system with the origin at the Earth's center of gravity and the z-axis directed toward the mean north pole as defined by the International Latitude Service. The x-z plane will coincide with the mean meridian of Greenwich. This definition is purely theoretical, since the Earth's center of gravity is not precisely known. For this reason, the positions given in this system will be in error and thus affect the satellite observations. A method to improve observing station positions from orbit analysis by the Minimum Variance Method will be presented later in the report.

Generally, the coordinates of an observing station are given as polar coordinates in a geodetic system. The coordinates are longitude, latitude and altitude. Longitude will be considered positive west for the station locations. Latitude is usually given as geodetic or geographic latitude. The two differ due to local gravity anomalies. Assuming that the given latitude is geodetic, we must obtain the geocentric latitude as a first step toward conversion to rectangular coordinates. It is important to note that the given polar coordinates are always associated with a specific ellipsoid, defined by the mean equatorial radius of the Earth, R_E, and flattening, f. The values for the adopted International Ellipsoid of Reference are R_E = 6,378,388 meters, and f = 1/297. More recent determinations have given R_E = 6,378,156 meters and f = 1/298.3.

The transformation from geodetic, ϑ, to geocentric latitude, ϕ, can be obtained by considering the geometry.

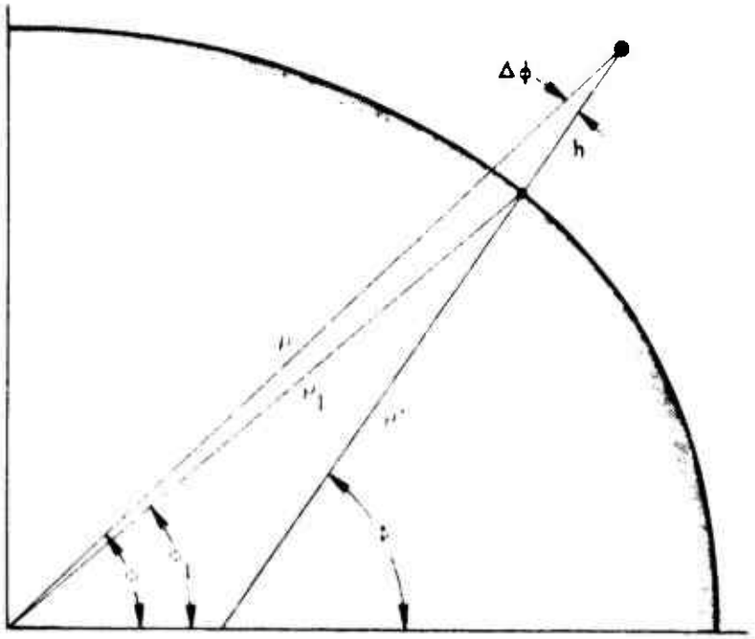

Fig. 1. Station Coordinates

Designating $f_E = (1 - f)^2$ we can write:

$$\tan\phi = \frac{(\rho' + h)\sin\vartheta}{h\cos\vartheta + \rho_1\cos\phi_1} \qquad (5)$$

where ρ_1, ρ' and h are expressed in units of R_E. Introducing an auxiliary function, C, defined by:

$$\rho_1\cos\phi_1 = C\cos\vartheta$$

$$C = \frac{1}{\cos \Phi \left(1 + f_E \tan^2 \Phi\right)^{1/2}}$$

and since $\rho_1 = f_E C$, we can express ϕ as function of Φ:

$$\tan \phi = \frac{h + f_E C}{h + C} \tan \Phi \tag{6}$$

Then the geocentric radius of the observing station is:

$$\rho = R_E (h + C) \frac{\cos \Phi}{\cos \phi} \tag{7}$$

The station coordinates in the Earth fixed terrestrial system are

$$x_s'' = \rho \cos \phi \cos (-\lambda) \tag{8a}$$

$$y_s'' = \rho \cos \phi \sin (-\lambda) \tag{8b}$$

$$z_s'' = \rho \sin \phi \tag{8c}$$

where λ is the station longitude (positive west).

These coordinates are in the terrestrial system as previously defined. The actual Earth's axis of rotation, however, does not coincide with the mean axis as defined by the International Latitude Service. It is moving about the mean pole in what is known as the polar motion. This motion has the effect of a small variation in latitude and meridian and must be considered in accurate calculations. The variations are regularly published by the International Latitude Service. The transformation of the station coordinates from the mean to the instantaneous system is accomplished by a simple transformation:

$$\begin{bmatrix} x_s' \\ y_s' \\ z_s' \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\underline{x} \\ 0 & 1 & \underline{y} \\ \underline{x} & -\underline{y} & 1 \end{bmatrix} \begin{bmatrix} x_s'' \\ y_s'' \\ z_s'' \end{bmatrix} \tag{9}$$

where $\underline{x}$ and $\underline{y}$ are the angular coordinates of the instantaneous pole, in radians.

D. STATION POSITION IN THE BASIC SYSTEM

So far the obtained position of the observing station is in a rotating Earth fixed-axes system. To relate it to the previously defined basic reference

system, which for all practical purposes is a true sidereal system, one more transformation must be performed.

The Earth fixed system and the basic system have the same z-axis (axis of rotation). The position of the observing station in the basic system, therefore, can be obtained by utilizing the true sidereal time, which is a function of the universal time (UT1). This will give the Greenwich hour angle of the vernal equinox as computed from the following equation:

$$\lambda_{GR} = 1.746647719 + 6.30038809863056\ d + 0.5064 \times 10^{-14}\ d^2 + \Delta\lambda \quad (10)$$

where λ_{GR} and $\Delta\lambda$ are in radians and d in Julian days from the epoch 1950 January 1, 0^h UT. As discussed in Chapter II, d must be expressed in UT1. The above equation is based on Newcomb's expression and corresponds to values for sidereal times published in the American Ephemeris and Nautical Almanac from 1960 on. The terms due to nutation are expressed by $\Delta\lambda$.

$$\begin{aligned}\Delta\lambda = &- 0.76700 \times 10^{-4} \sin (0.211408 - 0.00092422\ d) \\ &+ 0.929 \times 10^{-6} \sin (0.422816 - 0.00184844\ d) \\ &- 0.907 \times 10^{-6} \sin (2.247127 + 0.45994300\ d) \\ &- 0.5662 \times 10^{-5} \sin (9.776679 + 0.03440558\ d) \\ &+ 0.560 \times 10^{-6} \sin (6.248291 + 0.01720197\ d)\end{aligned} \quad (11)$$

The station coordinates in the basic system are then:

$$\begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = \begin{bmatrix} \cos\lambda_{GR} & -\sin\lambda_{GR} & 0 \\ \sin\lambda_{GR} & \cos\lambda_{GR} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'_s \\ y'_s \\ z'_s \end{bmatrix} \quad (12)$$

E. PRECESSION AND NUTATION

The basic system, as defined previously, is essentially a true sidereal system. As such, it follows the precessional and nutational motions of the Earth. The coordinate transformation from the reference mean equinox of 1950 January 1, 0^h UT to the mean equinox of date is accomplished by the following matrix:

$$P = \begin{bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix} \quad (13)$$

where:

$$X_x = 1 - 0.2226036 \times 10^{-12} d^2 - 0.267 \times 10^{-20} d^3$$

$$Y_x = -X_y = - 0.61190636 \times 10^{-6} d - 0.5067 \times 10^{-14} d^2 + 0.453 \times 10^{-19} d^3$$

$$Z_x = -X_z = - 0.26603997 \times 10^{-6} d + 0.1552 \times 10^{-14} d^2 + 0.197 \times 10^{-19} d^3$$

$$Y_y = 1 - 0.1872158 \times 10^{-12} d^2 - 0.308 \times 10^{-20} d^3$$

$$Y_z = Z_y = - 0.813972 \times 10^{-13} d^2 - 0.61 \times 10^{-21} d^3$$

$$Z_z = 1 - 0.353878 \times 10^{-13} d^2 + 0.41 \times 10^{-21} d^3$$

and d are Julian days since 1950 January 1, 0^h UT. To account for the nutation, a transformation from the mean equinox of date to the true equinox of date is accomplished by the following matrix:

$$N = \begin{bmatrix} 1 & -\Delta\mu & -\Delta\nu \\ \Delta\mu & 1 & -\Delta\epsilon \\ \Delta\nu & \Delta\epsilon & 1 \end{bmatrix} \qquad (14)$$

where $\Delta\mu$, $\Delta\nu$ and $\Delta\epsilon$ are the terms due to nutation in right ascension, declination and obliquity, respectively. They can be computed with sufficient accuracy from the following expressions

$$\Delta\mu = (-76.700 \text{ sin. } \alpha_1 + 0.929 \sin\alpha_2 - 0.907 \sin\alpha_3 - 5.662 \sin\alpha_4 + 0.560 \sin\alpha_5) \times 10^{-6}$$

$$\Delta\nu = (-33.909 \sin\alpha_1 + 0.400 \sin\alpha_2 - 0.390 \sin\alpha_3 - 2.437 \sin\alpha_4 + 0.241 \sin\alpha_5) \times 10^{-6}$$

$$\Delta\epsilon = (+44.654 \cos\alpha_1 - 0.438 \cos\alpha_2 + 0.428 \cos\alpha_3 + 2.676 \cos\alpha_4) \times 10^{-6}$$

where

$$\alpha_1 = 0.211408 - 0.00092422 \text{ d}$$

$$\alpha_2 = 0.422816 - 0.00184844 \text{ d}$$

$$\alpha_3 = 2.247127 + 0.45994300 \text{ d}$$

$$\alpha_4 = 9.776679 + 0.03440558 \text{ d}$$

$$\alpha_5 = 6.248291 + 0.01720197 \text{ d}$$

Only the most significant nutation terms are considered. However, they will give an accuracy of better than four meters on the Earth's surface.

The transformation of the rectangular coordinates between two basic or true sidereal systems of arbitrary dates can be accomplished by the following matrix multiplication:

$$x_{t_2} = N_2 P_2 P_1^T N_1^T x_{t_1} \tag{15}$$

where

$$x_{t_i} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{t_i} \text{ or } \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_{t_i} \qquad i = 1, 2$$

and T means the transpose of a matrix.

In a continuous, point-to-point transformation, only three multiplications must be performed at every transformation, since the transpose of the previous $N_2 P_2$ matrix is the new $P_1^T N_1^T$ matrix.

It is convenient to consider the precession and nutation matrices as sums of a unit matrix and a matrix with small elements.

Designating

$$P' = P - I$$

and

$$N' = N - I$$

where I is a unit matrix.

$$NP = (N' + I)(P' + I) = N'P' + P' + N' + I$$

Matrices N'P', P' and N' contain only small elements. Consequently, fewer significant figures need to be carried. The number of significant figures will increase for the diagonal elements only in the addition to the unit matrix. In some cases, it might be possible to neglect the second order term N'P' and thus the matrix multiplication could be avoided completely. Since the precessional and nutational motion is of the order of approximately 0".3 per day, the transformation need not be performed very often.

F. OBSERVATION REFERENCE SYSTEMS

Two types of observations will be considered: (1) observations referenced to a geodetic system and (2) observations referenced to a celestial system. Both types are necessarily topocentric systems. Types of observations referenced to a geodetic system will, generally, include elevation, azimuth, elevation and azimuth rate, range, range rate and range acceleration. Observations referenced to a celestial system are right ascension and declination. To perform the orbit analysis, the observations corresponding to the estimated orbit must be compared with the actual observations in a common reference system. The estimated observations from the given coordinates x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ are obtained as follows:

The satellite coordinates in a topocentric system where the x-axis points east, y-axis north and the z-axis completes a right-handed system, are:

$$\begin{bmatrix} x_T \\ y_T \\ z_T \end{bmatrix} = [S] \begin{bmatrix} x - x_s \\ y - y_s \\ z - z_s \end{bmatrix} \tag{16}$$

and

$$\begin{bmatrix} \dot{x}_T \\ \dot{y}_T \\ \dot{z}_T \end{bmatrix} = [S] \begin{bmatrix} \dot{x} + \omega_E y_s \\ \dot{y} - \omega_E x_s \\ \dot{z} \end{bmatrix} \tag{17}$$

where x_s, y_s, z_s are the station coordinates, and ω_E is the rate of rotation of the Earth. The matrix S is either

$$S = EM$$

or

$$S = M$$

depending on whether the transformation is performed to a geodetic or a geocentric system.

The transformation matrix, M, is:

$$M = \begin{bmatrix} -\sin(\lambda_{GR} - \lambda) & \cos(\lambda_{GR} - \lambda) & 0 \\ -\cos(\lambda_{GR} - \lambda)\sin\phi & -\sin(\lambda_{GR} - \lambda)\sin\phi & \cos\phi \\ \cos(\lambda_{GR} - \lambda)\cos\phi & \sin(\lambda_{GR} - \lambda)\cos\phi & \sin\phi \end{bmatrix} \tag{18}$$

where λ_{GR}, λ and ϕ are as defined previously.

The transformation from the geocentric to the geodetic system is performed by the matrix E

$$E \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\Delta\phi \\ 0 & \Delta\phi & 1 \end{bmatrix} \tag{19}$$

where $\Delta\phi$ = $\phi' - \phi$ and is a small angle.

The obtained coordinates x_T, y_T, z_T, $\dot{x}_T$, $\dot{y}_T$ and $\dot{z}_T$ of the satellite in the topocentric system allow us to compute the observations corresponding to the estimated orbit.

Observations of right ascension and declination are usually referred to a specific celestial system (see also Section VII B) defined by the equinox and equator at the beginning of a Besselian year. This will be discussed in more detail in Chapters IV and VII. At the present, we are concerned about the transformation of the satellite position from the basic system to a particular celestial system. This can be done by utilizing the previously given precession and nutation matrices. The coordinates of the satellite, as well as the observing station in the particular celestial system, will be:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_0 = [P_{ST}] \; [P]^T \; [N]^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{20}$$

where N is the transformation matrix due to nutation, P is the transformation matrix from the mean equinox of date to that of 1950 January 1, 0^h UT and P_{ST} is the transformation matrix to the mean equinox and equator of the celestial reference system. The elapsed days from 1950 January 1, 0^h UT to a standard epoch of a specific Besselian year can be obtained from the following formula:

$$d \quad -0.077 + 365.2422\,(BY - 1950.0) \tag{21}$$

where BY is the Besselian year; 1950.0 is a standard designation for the beginning of the Besselian year 1950.

If the celestial reference system is that of 1950.0, the matrix P_{ST} need not be included in most cases, since the epoch 1950 January 1, 0^h UT is very close to the standard epoch 1950.0.

Another method is to obtain the right ascension and declination of the satellite in the basic system and then to make the proper corrections for nutation and precession to obtain the right ascension and declination in the particular reference system of a standard epoch.

IV. SATELLITE OBSERVATIONS

A. CLASSIFICATION OF OBSERVATIONS

Satellite observations can be obtained generally by electronic or optical means. The electronic systems include pulse and Doppler radar and interferometer-type systems. Optical systems include visual and photographic observations. Recently, promising results have been obtained by laser systems which provide angular and range information.

The present discussion will be concerned with some pertinent aspects of electronic and optical systems providing observations in the form of elevation, azimuth, range, range rate, and right ascension and declination. In addition, observations of elevation and azimuth rates, and range acceleration will be considered.

B. ELECTRONIC OBSERVATIONS

From the analyst's point of view, the most important aspect of satellite observation is accuracy. Errors in electronic observations arise from several sources: errors in the electronic and mechanical system; errors due to refraction and aberration; and errors in the physical constants of the station location. Errors in the electronic system are due to causes such as frequency drifts, time delays, insufficient resolution, etc. Other errors are due to electromechanical systems. These errors can be considerable. Some error sources could be eliminated to a large extent if, instead of azimuth and elevation, their rates were measured. Thus, the precise knowledge of the true meridian and the geodetic vertical is not critical. The bias errors in the rate measurements should be smaller than in angle measurements.

Range, range rate, and range acceleration measurements rely entirely on the electronic system. Thus, a source of some highly unpredictable errors is eliminated. Moreover, the knowledge of the true north and vertical is not required. Consequently, these measurements can be much more accurate than the angle measurements.

One of the chief sources of error in electronic measurements is refraction. It is generally distinguished between tropospheric and ionospheric refraction. It has been shown (Ref. 6) that using frequencies in the kilomegacycle range, the ionospheric refraction is practically negligible and the tropospheric refraction can be calculated with fairly good accuracy, particularly if the local atmospheric conditions are taken into account.

The refraction error can be reduced to negligible proportions if a principle used in the Transit System (Ref. 7) is utilized. By simultaneously employing

two coherent frequencies, the actual refraction can be computed to a first order accuracy and the measurement corrected. In the present state of art, the theoretical accuracies obtainable for range and range rate measurements should be of the order of 5 m and 0.2 m/sec. However, the actual accuracies depend heavily on stringent calibration, maintenance and operation procedures, and, of course, the type of equipment.

C. OPTICAL OBSERVATIONS

Some of the most accurate satellite observations are obtained by optical techniques. Specifically, a network established and operated by the Smithsonian Astrophysical Observatory (Ref. 8) can provide observations with an accuracy of about ±2 seconds of arc.

Essentially, the precision method consists of photographing satellites against a star background. Since the star positions can be determined with high accuracy from star catalogs, the satellite right ascension and declination can be obtained with a comparable accuracy. The final accuracy of optical measurements, however, is dependent on the reduction method and may vary by orders of magnitude. Among the disadvantages of this method is that the precise reduction requires elaborate procedures and, therefore, the precise data are not immediately available. Another disadvantage is that the satellite can be photographed only at certain times when it is in sunlight and the observer in darkness. This can be minimized, if the satellite carries its own light source or by laser techniques. One of the chief advantages of this method is the elimination of many sources of error. The chief on-site requirement is the precise timing of the exposure. The actual computation of the measurements can be done under more exacting conditions.

D. OBSERVATION CORRECTIONS

The radar as well as optical observations must be corrected for aberration and refraction. Because of the relatively high velocities of satellites, the aberration effect will be significant in high precision measurements. This correction may amount to a few seconds of arc in angular measurements and several meters in range measurements. The correction is done on the basis of the satellite velocity vector and the speed of light.

A much more significant correction is required for refraction effects. In high precision measurements of right ascension and declination, obtained by the photographic method, the major part of the refraction is corrected indirectly. The remaining uncorrected part is the parallactic refraction which arises from the fact that the satellite is at a finite distance, while the stars are practically at infinity. Reference 8 gives the following expression for parallactic refraction correction:

$$\Delta\beta = -435''.0 \frac{\tan z}{r_T \cos z} \left(1 - e^{-0.1385\, r_T \cos z}\right) \qquad (22)$$

where z is the zenith distance and r_T the range.

The refraction correction for elevation, range and range rate in radar measurements can be computed by the following method (Refs. 9, 10, 11).

The troposphere is divided into m incremental layers and the change of the index of refraction in a layer is assumed linear.

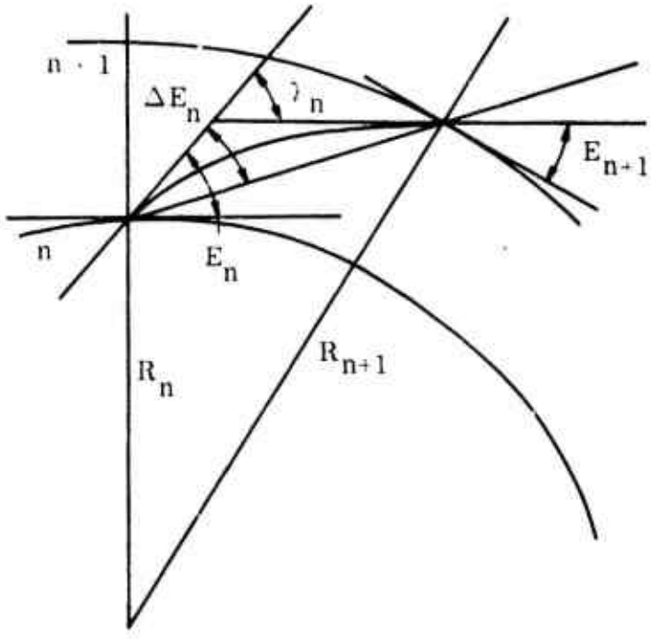

Fig. 2. Refraction Configuration

The average index of refraction as proposed by the NBS Central Radio Propagation Laboratory is given by the following expression

$$\eta_n = 1 + 0.000313 \exp(-0.14386\, h_n) \tag{23}$$

where h_n is the altitude in km.

Designating

$$N_n = \eta_n - 1$$

the total bending through the troposphere divided into m incremental layers is

$$\gamma = \sum_{n=0}^{n=m} \frac{2(N_n - N_{n+1})}{\tan E_n + \tan E_{n+1}} \text{ (rad)} \tag{24}$$

The elevation angle error is then

$$\Delta E = \frac{\gamma \tan E_m - (N_0 - N_m) + \gamma^2/2}{\gamma + \tan E_m - \tan E_0} \quad \text{(rad)} \tag{25}$$

and the range error is

$$\Delta r = \sum_{n=0}^{n\ m} \frac{(N_n + N_{n+1})(h_{n-1} - h_n)}{\sin E_n + \sin E_{n+1}} \tag{26}$$

The Doppler velocity can be corrected to a first order approximation by

$$\Delta \dot{r}_T = V \Delta \alpha_T \sin \phi \tag{27}$$

where ϕ is the angle between the line of sight and the velocity vector, $\vec{V}$. $\Delta\alpha_T$ is the angle between the line of sight and the velocity component in the direction of the ray path at the target. It can be determined from the expression

$$\Delta\alpha_T = \arccos\left[\frac{R_0}{R_m} \cos(E_0 - \Delta E)\right] - \arccos\left[\frac{\eta_0 R_0}{\eta_m R_m} \cos E_0\right] \tag{28}$$

The above expressions are valid for cases where the frequency is in the kilomegacycle range and the ionospheric refraction effects can be neglected.

V. APPLICATION OF THE MINIMUM VARIANCE METHOD

A. FORMULATION OF THE PROBLEM

The knowledge of an orbit implies the knowledge of the mathematical model of the dynamical system and certain constants associated with this system. The mathematical model can be developed by theory, but the constants must be determined by experiment. They are not observed directly, but can be determined knowing the mathematical relationships between the observed parameters and the constants. In our analysis, a part of the constants is represented by the so-called state variables, which in our case are the instantaneous orbital elements. The other part consists of the various physical constants of the dynamical and the observation system. Together they represent a generalized state vector.

The instantaneous orbital elements themselves are not constants. However, they are rigorously related to another set of orbital elements at some other time chosen at an epoch. The relationship between the two sets is defined by the mathematical model. Thus, the epochal set of the orbital elements, which are constants for the given system, completely determine the instantaneous orbital elements at any other time.

An approximate orbit can be obtained by assuming a simple inverse-square central force field. Various methods can be used for this purpose depending on the type of observations. With this starting orbit, an improved estimate of the orbital elements and/or the various physical constants can be obtained by more sophisticated methods. The application of the Minimum Variance Method for this purpose is the subject of this chapter.

There are three sources of errors associated with the orbit estimation process: incomplete representation of the dynamical system, errors in the generalized state vector and errors in the observations. The improvement of the dynamical model by statistical methods is beyond the scope of this report. Statistical knowledge about the errors in the generalized state vector and the observations, however, can be utilized in orbit improvement from observed data. The process, which is called statistical filtering, is applied to obtain a best estimate of the generalized state vector on the basis of the deviations of the actual observations from the estimated orbit. Once a good estimate of the orbit is obtained, past or future instantaneous orbital elements can be found by prediction and smoothing methods.

Orbit determination has always been a major problem in dynamical astronomy. Methods, notably, the Least Squares, have been in use for approximately a century. With the advent of the artificial Earth satellite, these methods were adapted for the new application. However, problems encountered in

practical space engineering applications often present quite different situations. A major factor is the electronic computer.

In dynamical astronomy, relatively few observations are obtained over an extended period of time. In many engineering applications, a large amount of observations are obtained within a comparatively short time period. The electronic computer offers immense possibilities, notably, its computing speed and storage capacity. However, new approaches may be necessary to take full advantage of these capabilities and also to comply with some of the numerical problems arising from the large amount of data. It is believed that the Minimum Variance Method offers great possibilities in many space engineering applications.

B. SOME PRACTICAL ASPECTS

The basic difference between the Minimum Variance Method and the Least Squares Method is that, in the Minimum Variance Method, the orbit is continuously updated on the basis of each new observation or set of observations. In the Least Squares Method, a single solution is obtained for all the observations simultaneously. In each case, an inversion of a certain order matrix is involved. The order of the matrix, in the least squares case, is determined by the number of the parameters being estimated. On the other hand, the order of the matrix to be inverted in the minimum variance case corresponds to the number of simultaneous observations. As a matter of fact, all the simultaneous observations need not be processed simultaneously, and, if desired, each can be handled separately. Thus the inversion becomes trivial. Since the inversion of high order matrices is not a simple matter, the advantages of the Minimum Variance Method are obvious.

Since the orbit is updated on each new set of observations, in many applications, the observations need not be stored in the computer, thus saving considerable storage space. The updating process is systematic for any number of observations, either too few or too many to obtain a deterministic solution for the generalized state vector. Also, the process can be interrupted at any time for any reason giving the optimum estimate at this point, and resumed at a later time. The Minimum Variance Method allows considerable flexibility in the method of application to suit the particular requirements.

In addition to the important practical aspects, the method is able to consider correlated measurement errors from one observation time to another.

C. LINEAR CONCEPTS

The theoretical development of the linear filtering theory used in this application is given by Kalman in Ref. 12 (see also Ref. 13). An application of the method is presented in Refs. 14 and 15. The following presentation will deal with the special case of satellite orbit determination and analysis.

The mathematical model of the dynamical system is represented by second-order differential equations of the form

$$\ddot{x}(t) = f\left[\dot{x}(t), x(t)\right] \tag{29}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

are the three position coordinates.

The individual terms of the right-hand side will be considered in a subsequent chapter. For the time being, it is sufficient to know that the equations are nonlinear. To enable the application of the linear filtering theory, the equations must be linearized and a solution obtained. The linearization can be accomplished by expanding the equations in a Taylor series about a reference trajectory and retaining only the first order terms. Thus

$$\ddot{x}(t) = f\left[\dot{x}_R(t), x_R(t)\right] + F(t)\,\Delta x(t) \tag{30}$$

where

$$F(t) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(t) & \dfrac{\partial f_1}{\partial x_2}(t) & \cdots & \dfrac{\partial f_1}{\partial \dot{x}_3}(t) \\ \dfrac{\partial f_2}{\partial x_1}(t) & \cdots & \cdots & \vdots \\ \dfrac{\partial f_3}{\partial x_1}(t) & \cdots & \cdots & \dfrac{\partial f_3}{\partial \dot{x}_3}(t) \end{bmatrix}$$

and

$$\Delta x(t) = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \\ \Delta \dot{x}_3 \end{bmatrix}$$

The differential equations for the reference trajectory are

$$\ddot{x}_R(t) = f\left[\dot{x}_R(t), x_R(t)\right] \tag{31}$$

If we subtract these from the perturbed equations, we obtain the linearized or perturbation equations

$$\Delta\ddot{x}(t) = F(t)\,\Delta x(t) \tag{32}$$

The three second-order equations can be redueed to six first-order equations by writing them in a standard form.

The fundamental solution to the obtained homogeneous equations is called the state transition matrix and designated Φ. The solution can be written in the form

$$\Delta x(t) = \Phi(t, t_o)\,\Delta x(t_o) \tag{33}$$

We now have obtained an algebraic matrix equation where the state transition matrix Φ relates linearly the vector Δx at time t_o to the vector Δx at time t. In a general case, a vector Δx^* will consist of all the parameters to be determined, and the matrix Φ^* will be of a corresponding order.

The Φ^* matrix possesses some important properties, which are summarized as follows:

$$\Phi^*(t, t) = I = \text{unit matrix} \tag{34}$$

$$\Phi^*(t_3, t_2)\,\Phi^*(t_2, t_1) = \Phi^*(t_3, t_1) \tag{35}$$

$$\Phi^{*-1}(t_2, t_1) = \Phi^*(t_1, t_2) \tag{36}$$

In practice, the matrix Φ for the six orbital elements can be obtained by several methods. In our case, it will be obtained on the assumption of an unperturbed orbit in an inverse-square central force field. Experience has shown that this is a good approximation for orbits where the atmospheric and gravitational perturbations are not significant. For very low altitude orbits, the approximation may be insufficient and corrections for atmospheric effects are necessary or the matrix must be obtained by other methods.

The equations that relate the observations to the instantaneous orbital elements and eonstants, in general, are also nonlinear. They can be linearized by a similar Taylor series expansion about a reference trajectory, and the equations can be written in the standard matrix form

$$\Delta y(t) = H^*(t)\,\Delta x^*(t) \tag{37}$$

where Δy is the deviation of the actual observations from the observations associated with the reference trajectory. H^* is the matrix of partial derivatives of the observations with respect to the instantaneous orbital elements and constants. The partial derivatives comprising the H^* matrix can be obtained and evaluated at the observation times in a straightforward manner.

The third important matrix used in the filtering equations is the covariance matrix. At any point in the filtering process, we have only an estimate of the instantaneous orbital elements and the constants. We can consider them as random scalars or components of a random vector. Thus the errors in the random variables, which are deviations from an expected or mean value, will also be random variables and as such will have a zero mean.

$$\epsilon\left\{\Delta x^*\right\} = 0 \tag{38}$$

The estimate of the individual random scalars is not known with the same accuracy. However, it can be described by the so-called variance which is defined as

$$\sigma^2 = \epsilon\left\{\left[\Delta x^* - \epsilon \Delta x^*\right]^2\right\} \tag{39}$$

where $\epsilon \Delta x^*$ is the expected value of the deviations, and is zero in our case. The standard deviation is defined as the square root of the variance.

The individual components of a random vector can be affected by the other components which is called correlation. Thus, instead of a single variance associated with a random scalar, a random vector has, in general, variances and covariances. The covariance matrix, P^*, is defined as

$$P^* = \text{cov}\left\{\Delta x^*,\ \Delta x^*\right\} = \epsilon\left\{\Delta x^*\ \Delta x^{*T}\right\} \tag{40}$$

where Δx^{*T} means the transpose of Δx^*.

The diagonal elements of this matrix are the variances and the off-diagonal elements are the covariances. If the components of the random vector are uncorrelated, the off-diagonal elements will be zero. If the covariance matrix, $P^*(t_o)$, is given at time, t_o, the covariance matrix, $P^*(t)$, at time, t, can be obtained by use of the state transition matrix Φ^*

$$\begin{aligned} P^*(t) &= \text{cov}\left\{\Delta x^*(t),\ \Delta x^*(t)\right\} = \epsilon\left\{\Delta x^*(t)\ \Delta x^{*T}(t)\right\} \\ &= \Phi^*(t, t_o)\ P^*(t_o)\ \Phi^{*T}(t, t_o) \end{aligned} \tag{41}$$

Similar considerations apply to the covariance matrix of the observation errors, Q.

D. FILTERING EQUATIONS

The statistical filtering theory is based on the assumption of a linear multi-dimensional dynamic system, which can be represented by the following model:

$$x^*(t) = \Phi^*(t, t_0)\, x^*(t_0) \tag{42}$$

$$y(t) = H^*(t)\, x^*(t_0) + u(t) \tag{43}$$

where $x^*(t)$ is an n-vector and represents the generalized state veetor, y(t) is a vector representing p independent measurements, and u(t) is a p-vector representing an independent gaussian random process or noise. The Φ^* and H^* matrices are n x n, and p x n matrices, respectively, which have been discussed before. Since the actual system is nonlinear, the above equations represent the linearized or perturbation equations. For the sake of simplicity, the perturbations are represented by x and y instead of Δx, Δy.

These perturbations are referred to the estimated orbit. Assuming that at time, t_{k+1}, an estimate of the state veetor is known based on the previous k observations, a better estimate can be obtained which includes the k + 1 observation. Under the linear assumptions, this ean be expressed as

$$\hat{x}^*(t_{k+1}) = \Phi^*(t_{k+1}, t_k)\, \hat{x}^*(t_k) + K^*(t_{k+1}) \left[y(t_{k+1}) - H^*(t_{k+1})\, \Phi^*(t_{k+1}, t_k)\, \hat{x}^*(t_k) \right] \tag{44}$$

where $\hat{x}$ designates the estimate of x. The first term on the right-hand side is simply the transfer of the state vector $\hat{x}^*(t_k)$ at time t_k to time t_{k+1} by the Φ^* matrix. Thus it represents the estimate of $\hat{x}^*$ at t_{k+1} based on the first k observations. The second term represents the contribution of the new observation at time t_{k+1}. The quantity in the brackets is the difference between the k th observation and the observation based on the estimated orbit. The matrix $K^*(t_{k+1})$ is a weighting matrix, sometimes ealled the optimum gains matrix, since it is obtained by optimizing a loss funetion.

As represented in the above equation, the assumed model of the dynamieal system is linear. However, in orbital analysis, it is advantageous to use the actual dynamical system represented by the nonlinear differential equations of motion to propagate the estimated state vector from one observation to the next. Also, the exaet equations can be used to obtain the observations associated with the estimated orbit. With these modifications, the equation reduces to

$$\hat{x}^*(t_{k+1}) = K^*(t_{k+1}) \left[y'(t_{k+1}) - \hat{y}'(t_{k+1}) \right] \tag{45}$$

where y' is the actual observation and $\hat{y}'$ is the observation based on the estimated orbit. Thus the linearized equations are used only to compute the weighting matrix and the covariance matrix. The advantages of this approach are that the process is less sensitive to errors in the initial conditions. The

nominal orbit, being continuously updated, does not deviate excessively from the true orbit. Consequently, the chance of violating the linearity assumptions is minimized.

Kalman (Ref. 12) obtains an optimal weighting matrix K* for an independent gaussian random process utilizing the Schmidt orthogonalization procedure in a multidimensional space. The optimal filter or weighting matrix is obtained utilizing the first and second-order statistics: the expectation of the state vector and the covariance matrix.

$$K^*(t_{k-1}) = P^*(t_{k-1})\, H^{*T}(t_{k-1}) \left[H^*(t_{k-1})\, P^*(t_{k-1})\, H^{*T}(t_{k+1})\right]^{-1} \tag{46}$$

where P* is the covariance matrix of the state vector and H* is a matrix of the partial derivatives of the observations with respect to the components of the state vector. The matrix $P^*(t_{k-1})$ is obtained transferring the $P^{*'}(t_k)$ matrix by means of the state transition matrix Φ^* and adding the error matrix Q*

$$P^*(t_{k-1}) = \Phi^*(t_{k-1}, t_k)\, P^{*'}(t_k)\, \Phi^{*T}(t_{k-1}, t_k) + Q^*(t_{k+1}) \tag{47}$$

Knowing the weighting matrix $K^*(t_{k-1})$, a new covariance matrix at time t_{k-1}, which includes the statistics of the new observation, can be obtained by the recursion equation

$$P^{*'}(t_{k-1}) = P^*(t_{k-1}) - K^*(t_{k-1})\, H^*(t_{k-1})\, P^*(t_{k+1}) \tag{48}$$

Since the state vector, generally, can be composed of many parameters, the matrices will be of a large order, affecting the numerical operations.

In Ref. 14, it is shown that a simplification of the matrix equations can be achieved if the errors in the observations are uncorrelated from one observation time to another. It is likely that for the same station, the observations will be correlated in this sense because of imperfect calibration, etc. In most cases, it will be sufficient to represent such errors by an algebraic function. Then the constants in this function can be regarded as bias errors, included in the state vector, and thus estimated along with the other constants. The remainder of the errors then can be considered as a Gaussian random vector and thus uncorrelated between two observation times. With this assumption, the equations reduce to

$$\hat{x}(t_{k+1}) = K(t_{k+1}) \left[y'(t_{k+1}) - \hat{y}'(t_{k+1})\right] \tag{49}$$

$$K(t_{k-1}) = P(t_{k-1})\, H^T(t_{k+1}) \left[H(t_{k-1})\, P(t_{k+1})\, H^T(t_{k+1}) + Q(t_{k+1})\right]^{-1} \tag{50}$$

$$P(t_{k-1}) = \Phi(t_{k-1}, t_k)\, P'(t_k)\, \Phi^T(t_{k+1}, t_k) \tag{51}$$

$$P(t_{k-1}) \quad P(t_{k-1}) - K(t_{k-1})\, H(t_{k-1})\, P(t_{k-1}) \tag{52}$$

where all the matrices, except the Q matrix, pertain to the system state vector only. The Q matrix represents the covariance matrix of the observations. If the state vector consists of the six orbital elements, the matrices will never be larger than 6 x 6. The matrix to be inverted will be of an order equal to the number of the simultaneous observations.

The inversion of the matrix is an important operation and deserves further analysis. If the observations are linearly independent, i.e., if none of the observed scalar random variables is a linear combination of the others, the matrix will be invertible whenever P and Q are positive definite. By virture of definition, the covariance matrices P and Q are positive definite, and thus the combination is invertible. The practical aspects of inverting a matrix, however, are quite different from those of pure theoretical considerations. It has been pointed out in Ref. 16 that the differential correction matrix tends to become singular as the time arc of the filtering process increases. The singularity is apparently also affected by the choice of the orbital elements. In the Least Squares Method, this will prevent a solution. In the Minimum Variance Method, the situation is somewhat different. First, as pointed out earlier, the order of the matrix to be inverted is equal to the number of simultaneous observations and, if desired, all the observations at a particular time need not be considered simultaneously. The inversion thus can be made trivial. Secondly, in most practical cases, the observations can be, indeed, considered independent, i.e., uncorrelated from one observation time to another. The Q matrix thus will be a diagonal matrix, representing the variances of the observations. As the filtering progresses, the orbit will become known with higher accuracy. In other words, the elements of matrix P will assume smaller numerical values. Consequently, the dominance of the Q matrix will become more pronounced, and since it is a diagonal matrix, no inversion problem will be encountered. Thus the choice of the orbital elements is not critical and may be made on the basis of other considerations. In the present case, the orbital elements are the position and velocity coordinates. In a program presently under development, no inversion difficulties have been encountered.

E. REJECTION OF OBSERVATIONS

The Minimum Variance Method is applicable to normally distributed random errors. It is inevitable that, in the observation data, there will be observations which for one reason or another will be greatly in error. As such they should not be included in the estimation, and some kind of criterion should be employed. This can be done as follows.

At any time during the filtering, an observation with a standard error must fall within a region represented by the sum of the covariance matrices of the estimated orbit and the observations.

$$R = H(t_k)\, P(t_k)\, H^T(t_k) + Q(t_k) \tag{53}$$

This covariance matrix represents an error ellipsoid of the observations (Ref. 17). Because of the covariance elements, the principal axes of this ellipsoid do not coincide with the axes system in which the observations are represented. Since the matrix is real and symmetric, it can be diagonalized by a similarity transformation. The diagonal elements thus would represent the excursions of the observations in the direction of the principal axes. However, the identity of the original observations would be lost.

Leaving the R matrix intact, we can partition it to a single element, and consider only the diagonal elements. These are called marginal deviations, and represent a case where the deviation of an individual observation is considered under the assumption that all the other deviations can be infinity. In a general case, instead of a hyperellipsoid, this will give a rectangular hyperparallelopiped. The square roots of the diagonal elements of matrix R thus will represent the standard deviations, σ, under these assumptions. Now, any actual observation which exceeds $n\sigma$, referred to the estimated orbit, can be rejected. In the case of a good representation of the mathematical model and knowledge of the observation errors, n could be 3 (3σ). It must be emphasized that the covariance matrices do not represent absolute numbers and, therefore, should be treated accordingly.

F. DETERMINATION OF CONSTANTS, TYPE 1

The previously given matrix equations can involve operations with large order matrices, in case constants are estimated simultaneously with the six orbital elements. However, recognizing the nature of the constants, certain simplifications can be introduced and the order of the matrices reduced.

There is a certain class of constants which are not functions of certain observations. In this case, the partial derivatives of the observations with respect to the constants are all zero. Constants of this type include the coefficients of the zonal harmonics, exospheric temperature of the atmosphere, Earth's mass, etc., in combination with observations of right ascension, declination, azimuth, elevation, range, range rate, azimuth rate and elevation rate.

Returning to the previously derived equations for $\hat{x}^*$, K^*, and P^*, we see that no difficulty is encountered writing $\hat{x}^*$ in a partitioned form

$$\hat{x}^* = \begin{bmatrix} \hat{x} \\ \hat{c} \\ \hat{n} \end{bmatrix} = \begin{bmatrix} K_x \\ K_c \\ K_n \end{bmatrix} [y' - \hat{y}'] \tag{54}$$

where $\hat{x}$, $\hat{c}$, $\hat{n}$, now denote estimates of perturbation vectors representing the six orbital elements, constants and measurements, respectively. Thus the vectors can be estimated independently. It remains to investigate the K* and P* matrices. We can write

$$P^*(t_{k+1}) = \Phi^*(t_{k+1}, t_k) P^*(t_k) \Phi^{*T}(t_{k+1}, t_k) + Q^*(t_{k+1}) \tag{55}$$

where $P^*(t_k)$ is in a partitioned form

$$P^*(t_k) = \begin{bmatrix} P'_x & P'_{xc} & P'_{xn} \\ P'^T_{xc} & P'_c & P'_{cn} \\ P'^T_{xn} & P'^T_{cn} & P'_n \end{bmatrix} \tag{56}$$

and P'_x, P'_c, P'_n are the covariance matrices of the orbital elements, constants, and observations, respectively. The submatrices with double subscripts designate the covariances between the three groups. Since a covariance matrix must be symmetric, the off-diagonal submatrices must be transposes of each other. The state transition matrix $\Phi^*(t_{k+1}, t_k)$ can be expressed as

$$\Phi^*(t_{k+1}, t_k) = \begin{bmatrix} \Phi_x & \Phi_{xc} & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{57}$$

where the Φ_x matrix is the state transition submatrix of the orbital elements. The state transition submatrix of the constants is a unit matrix I. Φ_{xc} is a submatrix relating the state of the elements at time t_{k+1}, as affected by the state of the constants at t_k. The other submatrices are zero because: (1) there is no correlation between the observation errors at time t_k, which are assumed Gaussian random errors, and the orbital elements or constants at time t_{k+1}; (2) the constants are not affected by errors in the orbital elements; and (3) the observation errors are uncorrelated from one observation time to another.

The covariance matrix $Q^*(t_{k+1})$ is simply

$$Q^*(t_{k+1}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q' \end{bmatrix} \tag{58}$$

and represents the observation errors.

Now, performing the required operations we obtain

$$P^*(t_{k-1}) = \begin{bmatrix} \left[(\Phi_x P'_x + \Phi_{xc} P'^T_{xc})\ \Phi_x^T + (\Phi_x P'_{xc} + \Phi_{xc} P'_c)\ \Phi_{xc}^T\right] & \left[\Phi_x P'_{xc} + \Phi_{xc} P'_c\right] & 0 \\ \left[P'^T_{xc}\ \Phi_x^T + P'_c\ \Phi_{xc}^T\right] & P'_c & 0 \\ 0 & 0 & Q' \end{bmatrix}$$

$$= \begin{bmatrix} P_x & P_{xc} & 0 \\ P_{xc}^T & P_c & 0 \\ 0 & 0 & Q' \end{bmatrix} \tag{59}$$

Thus the transfer of the covariance matrix $P^{*\prime}(t_k)$ can be accomplished in parts

$$P_{xc} = \Phi_x P'_{xc} + \Phi_{xc} P'_c \tag{60}$$

$$P_x = (\Phi_x P'_x + \Phi_{xc} P'^T_{xc})\ \Phi_x^T + P_{xc}\ \Phi_{xc}^T \tag{61}$$

$$P_c = P'_c \tag{62}$$

Note that $P_c^T = P_c$.

In the expression for $K^*(t_{k+1})$, the quantity in brackets is

$$H^*\ P^*\ H^{*T} = \begin{bmatrix} H_x & H_c & H_n \end{bmatrix} \begin{bmatrix} P_x & P_{xc} & 0 \\ P_{xc}^T & P_c & 0 \\ 0 & 0 & Q' \end{bmatrix} \begin{bmatrix} H_x^T \\ H_c^T \\ H_n^T \end{bmatrix}$$

In our case $H_c = 0$, and performing the matrix multiplication

$$H^*\ P^*\ H^{*T} = H_x\ P_x\ H_x^T + H_n Q' H_n^T = R$$

Thus the weighting matrix is

$$K^* = P^*\ H^{*T}\ R^{-1} = \begin{bmatrix} P_x H_x^T \\ P_{xc}^T H_x^T \\ Q' H_n^T \end{bmatrix} \left[R\right]^{-1}$$

or

$$K_x = P_x\ H_x^T\ R^{-1} \tag{63}$$

$$K_c = P_{xc}^T\ H_x^T\ R^{-1} \tag{64}$$

Performing the operations in the recursion equation we obtain

$$P^{*\prime} = P^* - K^* H^* P^* =$$

$$\begin{bmatrix} P_x & P_{xc} & 0 \\ P_{xc}^T & P_c & 0 \\ 0 & 0 & Q' \end{bmatrix} - \begin{bmatrix} P_x H_x^T R^{-1} \\ P_{xc}^T H_x^T R^{-1} \\ Q' H_n^T R^{-1} \end{bmatrix} \begin{bmatrix} H_x & 0 & H_n \end{bmatrix} \begin{bmatrix} P_x & P_{xc} & 0 \\ P_{xc}^T & P_c & 0 \\ 0 & 0 & Q' \end{bmatrix} =$$

$$\begin{bmatrix} P_x & P_{xc} & 0 \\ P_{xc}^T & P_c & 0 \\ 0 & 0 & Q' \end{bmatrix} - \begin{bmatrix} (P_x H_x^T R^{-1} H_x P_x) & (P_x H_x^T R^{-1} H_x P_{xc}) & (P_x H_x^T R^{-1} H_n Q') \\ (P_{xc}^T H_x^T R^{-1} H_x P_x) & (P_{xc}^T H_x^T R^{-1} H_x P_{xc}) & (P_{xc} H_x^T R^{-1} H_n Q') \\ (Q' H_n^T R^{-1} H_x P_x) & (Q' H_n^T R^{-1} H_x P_{xc}) & (Q' H_n^T R^{-1} H_n Q') \end{bmatrix}$$

Now the updating of the P* matrix can be done by parts. Thus the covariance matrix of the orbital elements is

$$P_x' = P_x - P_x H_x^T R^{-1} H_x P_x = P_x - K_x H_x P_x \tag{65}$$

The updating of the covariance matrix of the constants is accomplished by the equation

$$P_c' = P_c - P_{xc}^T H_x^T R^{-1} H_x P_{xc} = P_c - K_c H_x P_{xc} \tag{66}$$

It should be noted that the second term on the right-hand side is strongly dependent on the correlation between the constants and the orbital elements. If the correlation is weak, the updating will be very ineffective, as will be the estimation of the constants.

Similarly, the covariance submatrix P_{xc} will be updated as follows

$$P_{xc}' = P_{xc} - P_x H_x^T R^{-1} H_x P_{xc} = P_{xc} - K_x H_x P_{xc} \tag{67}$$

The other submatrices P_{xn} and P_{cn} need not be updated because they will vanish in the transformation.

G. DETERMINATION OF CONSTANTS, TYPE 2

A similar simplification of the filtering equations can be accomplished if the constants to be estimated are not related to the orbital elements, i.e., they are not included in the equations of motion. Constants of this type include the coordinates of observing stations and bias errors in the observations.

Again we can write the covariance matrix in a partitioned form:

$$P^{*\prime}(t_k) = \begin{bmatrix} P'_x & P'_{xc} & P'_{xn} \\ P'^T_{xc} & P'_c & P'_{cn} \\ P'^T_{xn} & P'^T_{cn} & P'_n \end{bmatrix} \tag{68}$$

where the submatrices are defined previously.

The state transition matrix now will be

$$\Phi^*(t_{k+1}, t_k) = \begin{bmatrix} \Phi_x & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{69}$$

Regarding the correlation from one observation time to another we have made the same assumptions as in the previous case with the following modification. The observation errors now are assumed correlated and represented by an error function. The constants in this function are represented in the covariance matrix P_c and estimated together with the other constants of this type. The uncompensated observation errors are considered Gaussian random errors and as such uncorrelated from one observation time to another. The transition submatrix Φ_{xc} will be zero because the constants do not enter into the equations of motion. The covariance matrix Q* is the same as previously.

Performing the required operations we obtain

$$P^*(t_{k+1}) = \begin{bmatrix} \Phi_x P'_x \Phi_x^T & \Phi_x P'_{xc} & 0 \\ P'^T_{xc} \Phi_x^T & P'_c & 0 \\ 0 & 0 & Q' \end{bmatrix} = \begin{bmatrix} P_x & P_{xc} & 0 \\ P_{xc}^T & P_c & 0 \\ 0 & 0 & Q' \end{bmatrix} \tag{70}$$

and the transfer of the covariance matrix can be done in parts

$$P_x = \Phi_x P'_x \Phi_x^T \tag{71}$$

$$P_{xc} = \Phi_x P'_{xc} \tag{72}$$

$$P_c = P'_c \tag{73}$$

The weighting matrix K* can be obtained, first, considering the expression in the brackets.

$$H^* P^* H^{*T} = (H_x P_x + H_c P_{xc}^T) H_x^T + (H_x P_{xc} + H_c P_c) H_c^T + H_n Q' H_n^T = R$$

The weighting matrix is then

$$K^* = P^* H^{*T} R^{-1} = \begin{bmatrix} P_x H_x^T + P_{xc} H_c^T \\ P_{xc}^T H_x^T + P_c H_c^T \\ Q' H_n^T \end{bmatrix} \left[R \right]^{-1}$$

or

$$K_x = (P_x H_x^T + P_{xc} H_c^T) R^{-1} \tag{74}$$

$$K_c = (P_{xc}^T H_x^T + P_c H_c^T) R^{-1} \tag{75}$$

The recursion equation then can be obtained as follows:

$$P^{*\prime} = P^* - K^* H^* P^* = \begin{bmatrix} P_x & P_{xc} & 0 \\ P_{xc}^T & P_c & 0 \\ 0 & 0 & Q' \end{bmatrix} - \begin{bmatrix} K_x \\ K_c \\ K_n \end{bmatrix} \begin{bmatrix} H_x H_c H_n \end{bmatrix} \begin{bmatrix} P_x & P_{xc} & 0 \\ P_{xc}^T & P_c & 0 \\ 0 & 0 & Q' \end{bmatrix}$$

Which gives after performing the required operations

$$P_x' = P_x - K_x(H_x P_x + H_c P_{xc}^T) \tag{76}$$

$$P_{xc}' = P_{xc} - K_x(H_x P_{xc} + H_c P_c) \tag{77}$$

$$P_c' = P_c - K_c(H_x P_{xc} + H_c P_c) \tag{78}$$

Thus the order of the matrices is reduced, and, for the diagonal submatrices, it will be equal to the number of the orbital elements or constants, respectively. For the covariance matrix P*, the submatrices forming the rows and columns are simply transposes of each other. The submatrices P_{xn}' and P_{cn}' and their transposes in P*' need not be computed since they will vanish in the transformation. In either case, the maximum order of matrix R, which must be inverted, is equal to the number of simultaneous observations.

The equations as derived in this and the previous section are for a simultaneous estimation of the orbital elements and constants. Obviously, the whole process can be separated in two parts. First, the constants may be assumed known and a best fitting orbit determined as in a normal orbit determination

routine. The residuals then can be attributed to the constants and the filtering repeated considering only the equations pertaining to the constants. The correlation between the orbital elements and the constants thus will be ignored but the estimation process will be simplified.

VI. STATE TRANSITION MATRIX

A. FORMULATION OF THE PROBLEM

One of the matrices in the filtering equations is the state transition matrix of the six orbital elements. This matrix can be obtained for the exact mathematical model by the so-called secant technique, i.e., perturbing the elements one at a time and obtaining a solution by integration. For observations close together, and using the Minimum Variance Method in a point-to-point mode as outlined previously, this can be a very efficient method. If the observations are far apart, the method is very time consuming.

The matrices used in the filtering equations are not arrays of absolute numbers, and so do not require absolute precision. Therefore, a good approximation of the actual dynamical system is permissible. Experience has shown that such good approximation is a simple inverse-square central force field. The resulting orbits are a circle, ellipse, parabola or hyperbola, depending on the eccentricity. Even with this approximation, the analytical solution has presented considerable challenge. As a result, many analytical methods have been developed and published. A pure, closed form analytical solution, however, is not always the most satisfactory for the electronic computer. The computer is most efficient for repeated solutions of simple arithmetic equations, which save storage space and computation time. The method that follows has been developed with these considerations in mind.

The problem can be stated as follows. Given the six orbital elements in the form of rectangular coordinates x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$, at time t_1, find a state transition matrix which is defined as one relating small perturbations of the state at time t_1 to the resulting perturbations at time t_2. A fundamental solution will be obtained, first, for an elliptic orbit, and then extended to circular, parabolic and hyperbolic orbits. Since orbits in a central force field are planar, the solution can be obtained in three steps: (1) in-plane perturbations; (2) out-of-plane perturbations; (3) transformation to the original axes system.

B. ELLIPTIC ORBITS

The direction cosines of an axes system in which the x_ω-axis is directed toward the point on the orbit at time t_1, y_ω-axis is in the orbital plane such that $\dot{y}_\omega > 0$, and z_ω-axis completes a right hand system are, first for the x_ω-axis

$$\xi_1 = \frac{x_1}{r_1} \quad , \quad \eta_1 = \frac{y_1}{r_1} \quad , \quad \zeta_1 = \frac{z_1}{r_1} \qquad (79a, b, c)$$

The direction cosines for the z_ω-axis are obtained by taking the vector product $\vec{r_1} \times \vec{v_1}$. Then

$$\xi_3 = \frac{y_1 \dot{z}_1 - z_1 \dot{y}_1}{d_1} \tag{80a}$$

$$\eta_3 = \frac{z_1 \dot{x}_1 - x_1 \dot{z}_1}{d_1} \tag{80b}$$

$$\zeta_3 = \frac{x_1 \dot{y}_1 - y_1 \dot{x}_1}{d_1} \tag{80c}$$

where

$$d_1 = \left[(y_1 \dot{z}_1 - z_1 \dot{y}_1)^2 + (z_1 \dot{x}_1 - x_1 \dot{z}_1)^2 + (x_1 \dot{y}_1 - y_1 \dot{x}_1)^2\right]^{1/2}$$

Similarly, the direction cosines of the y_ω-axis are obtained from the vector product of the unit vectors in the direction of z_ω-and x_ω-axes.

$$\xi_2 = \eta_3 \zeta_1 - \zeta_3 \eta_1 \tag{81a}$$

$$\eta_2 = \zeta_3 \xi_1 - \xi_3 \zeta_1 \tag{81b}$$

$$\zeta_2 = \xi_3 \eta_1 - \eta_3 \xi_1 \tag{81c}$$

The velocity components in the new planar axes system are

$$\begin{bmatrix} \dot{x}_{\omega_1} \\ \dot{y}_{\omega_1} \end{bmatrix} = \begin{bmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{bmatrix} \tag{82}$$

also $$y_{\omega_1} = z_{\omega_1} = \dot{z}_{\omega_1} = 0 \tag{83}$$

We can now compute a set of orbital elements which define the planar orbit (see Appendix).

$$A = 1 - \frac{x_{-1}\,\dot{y}_{-1}^2}{\mu} \tag{84}$$

$$B = \frac{x_{-1}\,\dot{x}_{-1}\,\dot{y}_{-1}}{\mu} \tag{85}$$

$$C = \frac{x_{-1}^2\,\dot{y}_{-1}^2}{\mu} \tag{86}$$

A and B give eccentricity

$$e = \left(A^2 + B^2\right)^{1/2} \tag{87}$$

Rotating the planar axes system so that the $\bar{x}$-axis points toward the pericenter, the direction cosines of the new system are

$$\xi_a = -\frac{A}{e} \quad , \quad \eta_a = -\frac{B}{e} \tag{88a, b}$$

and the normalized coordinates of the initial point in this axes system are

$$\bar{x}_1 = \frac{x_{-1}}{C}\,\xi_a \quad , \quad \bar{y}_1 = -\frac{x_{-1}}{C}\,\eta_a \tag{89a, b}$$

We can now find the complementary angle

$$\theta_1 = \arctan\left[\frac{e + (1 - e^2)\,\bar{x}_1}{(1 - e^2)^{1/2}\,\bar{y}_1}\right] \tag{90}$$

The eccentric anomaly is

$$E_1 = \frac{\pi}{2} - \theta_1 \qquad \text{if } \bar{y}_1 \geq 0 \tag{91a}$$

$$E_1 = \frac{3}{2}\,\pi - \theta_1 \qquad \text{if } \bar{y}_1 < 0 \tag{91b}$$

The mean anomaly for the initial point is from Kepler's equation

$$M_1 = E_i - e \sin E_1 \tag{92}$$

Now the eeeentric anomaly for the final point ean be obtained by an iteration method. For a first approximation

$$E_2 = M_1 + \frac{\Delta t}{K_c} \tag{93}$$

where

$$K_c = \left[\frac{C^3}{(1 - e^2)^3 \mu}\right]^{1/2} \qquad \text{and } \Delta t = t_2 - t_1$$

and the solution ean be obtained to the desired degree of aeeuracy by sueeessively computing

$$M_2 = E_2 - e \sin E_2 \tag{94}$$

$$\Delta t_o = K_c \left(M_2 - M_1\right) \tag{95}$$

$$E_2 = E_{2_o} + \frac{\Delta t - \Delta t_o}{K_c \left(1 - e \cos E_{2_o}\right)} \tag{96}$$

where the subscript o designates the previous estimate.

The position and velocity of the final point in the planar axes system ean be obtained by solving the following set of equations

$$\bar{x}_2 = \frac{\cos E_2 - e}{1 - e^2} \tag{97a}$$

$$\bar{y}_2 = \frac{\sin E_2}{(1 - e^2)^{1/2}} \tag{97b}$$

$$x_2 = \frac{\left(\bar{x}_2 \xi_a - \bar{y}_2 \eta_a\right) C}{\xi_a^2 + \eta_a^2} \tag{98a}$$

$$y_{\omega_2} = \frac{\left(\bar{x}_2 \eta_a + \bar{y}_2 \xi_a\right) C}{\xi_a^2 + \eta_a^2} \tag{98b}$$

$$r_2 = \left(x_{\omega_2}^2 + y_{\omega_2}^2\right)^{1/2} \tag{99}$$

$$\dot{x}_{\omega_2} = \frac{B\, r_2 - y_{\omega_2}}{r_2} \left(\frac{\mu}{C}\right)^{1/2} \tag{100a}$$

$$\dot{y}_{\omega_2} = -\frac{A\, r_2 - x_{\omega_2}}{r_2} \left(\frac{\mu}{C}\right)^{1/2} \tag{100b}$$

This process must be repeated either four or eight times, perturbing successively x_{ω_1}, y_{ω_1}, $\dot{x}_{\omega_1}$, $\dot{y}_{\omega_1}$, and starting with Eq. (84). Since the perturbation of y_{ω_1} means the reorientation of the axes system, the perturbed velocities $\dot{x}_{\omega_1}$ and $\dot{y}_{\omega_1}$ must be transformed in the new axes system, before solving for the elements A, B, C, by the matrix

$$\begin{bmatrix} \dfrac{r_1}{x_{\omega_1}} & \dfrac{\Delta y_{\omega_1}}{x_{\omega_1}} \\ -\dfrac{\Delta y_{\omega_1}}{x_{\omega_1}} & \dfrac{r_1}{x_{\omega_1}} \end{bmatrix} \tag{101}$$

where Δy_{ω_1} is the perturbation and

$$x_{\omega_1} = \left(r_1^2 + \Delta y_{\omega_1}^2\right)^{1/2}$$

After obtaining the final values x_{ω_2}, y_{ω_2}, $\dot{x}_{\omega_2}$, $\dot{y}_{\omega_2}$ for this case, they must be transformed into the original system by multiplying by the transpose of this matrix.

Each perturbation will give the partials of x_{ω_2}, y_{ω_2}, $\dot{x}_{\omega_2}$, $\dot{y}_{\omega_2}$ with respect to the particular perturbation. Thus

$$\phi_{11} = \frac{\Delta x_{\omega_2}}{\Delta x_{\omega_1}} = \frac{x_{\omega_2} - x_{\omega_{2n}}}{\Delta x_{\omega_1}}, \text{ etc.}$$

where $x_{\omega_{2n}}$ is the nominal value of x_{ω_2}.

A state transition matrix for a planar orbit in the $x_\omega - y_\omega - z_\omega$ axes system can be written as

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & 0 & \phi_{14} & \phi_{15} & 0 \\ \phi_{21} & \phi_{22} & 0 & \phi_{24} & \phi_{25} & 0 \\ 0 & 0 & \phi_{33} & 0 & 0 & \phi_{36} \\ \phi_{41} & \phi_{42} & 0 & \phi_{44} & \phi_{45} & 0 \\ \phi_{51} & \phi_{52} & 0 & \phi_{54} & \phi_{55} & 0 \\ 0 & 0 & \phi_{63} & 0 & 0 & \phi_{66} \end{bmatrix} \tag{102}$$

The first, second, fourth, and fifth columns of the matrix have been obtained by the perturbation technique. The indicated elements in these columns must be zero because, for a planar orbit, the in-plane perturbations cannot cause out-of-plane deviations. It remains to determine the third and the sixth column which is due to the out-of-plane or Δz_ω and $\Delta \dot{z}_\omega$ perturbations. The main effect of these perturbations is to tilt the resulting orbit with respect to the original orbital plane. For small perturbations of z_ω, and $\dot{z}_\omega$, the tilting angle and the increment in total velocity is small. To a first order approximation, assuming that the cosine of small angles is equal to one, they will have no effect on the x_ω and y_ω coordinates. Thus the indicated elements in columns three and six are assumed zero.

Considering a new axes system with the $\bar{x}_{\omega}$-axis displaced by Δz_{ω_1} and the new orbital plane determined by the new velocity vector $\vec{v}_{\omega_1} - \Delta\dot{\vec{z}}_{\omega_1}$ we can obtain the direction cosines of the new $\bar{z}_{\omega}$-axis neglecting higher order terms

$$\bar{\xi} = -\frac{\Delta z_{\omega_1}}{x_{\omega_1}}$$

$$\bar{\eta} = \frac{\dot{x}_{\omega_1}}{x_{\omega_1}\dot{y}_{\omega_1}} \Delta z_{\omega_1} - \frac{\Delta \dot{z}_{\omega_1}}{\dot{y}_{\omega_1}}$$

$$\bar{\zeta} = 1$$

Because of orthogonality, a projection of coordinates in the new system on the original z_{ω}-axis will be

$$z_{\omega_2} = \frac{\Delta z_{\omega_1}}{x_{\omega_1}} \bar{x}_{\omega_2} + \left(\frac{\Delta \dot{z}_{\omega_1}}{\dot{y}_{\omega_1}} - \frac{\dot{x}_{\omega_1} \Delta z_{\omega_1}}{x_{\omega_1}\dot{y}_{\omega_1}} \right) \bar{y}_{\omega_2} - \bar{z}_{\omega_2}$$

We previously established that, because of the small angles, $\bar{x}_{\omega_2} = x_{\omega_2}$ and $\bar{y}_{\omega_2} = y_{\omega_2}$ to a first order approximation. Now if $\bar{z}_{\omega_2} = 0$ (for a planar orbit)

$$\Delta z_{\omega_2} = \left(\frac{x_{\omega_2}}{x_{\omega_1}} - \frac{\dot{x}_{\omega_1} y_{\omega_2}}{x_{\omega_1}\dot{y}_{\omega_1}} \right) \Delta z_{\omega_1} + \frac{y_{\omega_2}}{\dot{y}_{\omega_1}} \Delta \dot{z}_{\omega_1}$$

By comparing this equation to the third row in matrix ϕ, we obtain

$$\phi_{33} = \frac{\Delta z_{\omega_2}}{\Delta z_{\omega_1}} = \frac{x_{\omega_2}}{x_{\omega_1}} - \frac{\dot{x}_{\omega_1} y_{\omega_2}}{x_{\omega_1}\dot{y}_{\omega_1}} \tag{103}$$

and

$$\phi_{36} = \frac{\Delta z_{\omega_2}}{\Delta \dot{z}_{\omega_1}} = \frac{y_{\omega_2}}{\dot{y}_{\omega_1}} \tag{104}$$

By a similar process we obtain

$$\phi_{63} = \frac{\Delta\dot{z}_{\sim 2}}{\Delta z_{\sim 1}} = \frac{\dot{x}_{\sim 2}}{x_{\sim 1}} - \frac{\dot{x}_{\sim 1}\,\dot{y}_{\sim 2}}{x_{\sim 1}\,\dot{y}_{\sim 1}} \tag{105}$$

$$\phi_{66} = \frac{\Delta\dot{z}_{\sim 2}}{\Delta\dot{z}_{\sim 1}} = \frac{\dot{y}_{\sim 2}}{\dot{y}_{\sim 1}} \tag{106}$$

Thus all the elements in matrix ϕ are known. Forming a matrix X from the previously computed direction cosines

$$X = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_3 & 0 & 0 & 0 \\ \zeta_1 & \zeta_2 & \zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 0 & 0 & \eta_1 & \eta_2 & \eta_3 \\ 0 & 0 & 0 & \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix} \tag{107}$$

We can now obtain the state transition matrix in the original rectangular coordinate system

$$\Phi = X\,\phi\,X^{T} \tag{108}$$

The general procedure is valid for all eccentricities with exception of the particular computations as outlined next.

C. CIRCULAR ORBITS

In the expressions for the direction cosines ξ_a and η_a, the eccentricity e appears in the denominator. However, because e is obtained from $(A^2 + B^2)^{1/2}$, no numerical difficulties will be encountered for small eccentricities as long as a sufficient number of significant figures are carried. Obviously, circularity is a relative matter. Thus at some point the orbit can be assumed circular. Since a circular orbit has no pericenter, we can assume an orientation of the $\bar{x}$-axis to coincide with the initial point. Therefore, the direction cosines are

$$\xi_a = -1, \quad \eta_a = 0 \tag{109a, b}$$

and the final eccentric anomaly is

$$E_2 = \frac{\Delta t}{K_c} \tag{110}$$

The rest of the solution is the same as for the elliptic case.

D. PARABOLIC ORBITS

A parabolic orbit has an eccentricity, $e = 1$. The chances of a pure parabolic or a pure circular orbit occurring in computations are low and, as pointed out previously, in fact, is a relative matter. In the assumed rectangular coordinate system, the solution for a parabolic orbit can be obtained in a closed form. First, the normalized area swept out by a radius vector from pericenter to the initial point is (see Appendix)

$$A_1 = \frac{\bar{y}_1^3}{3} + \frac{\bar{y}_1 \bar{x}_1}{2} \tag{111}$$

and the area swept out to the final point is

$$A_2 = A_1 + \frac{\Delta t}{K_c} \tag{112}$$

where

$$K_c = 2\left(\frac{C^3}{\mu}\right)^{1/2}$$

from which the coordinates of the final point are

$$\bar{y}_2 = \left[6A_2 + \left(36A_2^2 + 1\right)^{1/2}\right]^{1/3} + \left[6A_2 - \left(36A_2^2 + 1\right)^{1/2}\right]^{1/3} \tag{113a}$$

$$\bar{x}_2 = \frac{1}{2}\left(1 - \bar{y}_2^2\right) \tag{113b}$$

The rest of the solution is the same as for the elliptic case.

E. HYPERBOLIC ORBITS

Hyperbolic and elliptic orbits are the most important orbits. A hyperbolic orbit is one with eccentricity, $e > 1$. The solution for a hyperbolic orbit must be obtained by iteration. We can write the normalized area swept out by the radius vector from pericenter to the initial point

$$A_1 = e\bar{y}_1 + \frac{1}{(e^2-1)^{1/2}} \ln B_1 \tag{114}$$

where

$$B_1 = e^2 - (e^2-1)\,\bar{x}_1 - (e^2-1)^{1/2}\,\bar{y}_1$$

For a first estimate of the final point we assume a parabolic orbit. The estimate of the swept-out area to the final point is

$$A_2 = A_1 + \frac{\Delta t}{2(e^2-1)K_c} \tag{115}$$

where

$$K_c = \frac{1}{e^2-1}\left(\frac{C^3}{\mu}\right)^{1/2}$$

The estimated $\bar{y}_2$ and $\bar{x}_2$ are obtained from Eqs. (113a), (113b) and an improved A_2 obtained from

$$A_2 = e\bar{y}_2 + \frac{1}{(e^2-1)^{1/2}} \ln B_2 \tag{116}$$

where

$$B_2 = e - (e^2-1)\bar{x}_2 - (e^2-1)^{1/2}\,\bar{y}_2$$

The time corresponding to the estimated final point is

$$\Delta t_o = K_c\left(A_2 - A_1\right) \tag{117}$$

The iteration is done on $\bar{y}_2$ successively solving Eqs. (116), (117), (118), and (119) until the desired degree of accuracy is reached.

$$\bar{y}_2 = \bar{y}_{2_o} + \frac{B_{2_o}\left[(e^2-1)\,\bar{x}_{2_o} - e\right](\Delta t - \Delta t_o)}{K_c\left\{\left(B_{2_o}\,e-1\right)\left[(e^2-1)\bar{x}_{2_o} - e\right] - (e^2-1)^{1/2}\,\bar{y}_{2_o}\right\}} \tag{118}$$

$$\bar{x}_2 = \frac{e - \left[1 + (e^2-1)\bar{y}_2^2\right]^{1/2}}{e^2-1} \tag{119}$$

The remaining solution is the same as for the elliptic case.

Experience has shown that the initial estimates of $\bar{x}_2$ and $\bar{y}_2$ in the hyperbolic case are unimportant and even grossly inaccurate estimates will give rapid convergenee using thc above method. Therefore, the use of the parabolic solution for the initial estimates is justified.

VII. EQUATIONS OF OBSERVATIONS

A. REQUIREMENTS FOR ANALYSIS

To obtain the residuals or differences between the observed and estimated observations, it is necessary to compute the associated observations from the estimated orbit. In addition, the filtering equations require the H-matrix which has been defined as a matrix of the partial derivatives of the observations with respect to the state variables. As shown previously, the H-matrix can be partitioned to simplify the matrix operations. It was also noted that for a large class of constants the H_c-submatrix was either zero or a unit matrix. The relationship between the observations and the orbital elements as well as the associated H_x-matrix remains to be determined.

B. ANALYTIC EXPRESSIONS

To obtain the expressions relating elevation and azimuth to the orbital elements, we will utilize the equations given in Section III-F. Knowing the satellite coordinates x_T, y_T, z_T in the topocentric axes system as defined previously, elevation and azimuth can be expressed as

$$E = \arctan\left[\frac{z_T}{\left(x_T^2 + y_T^2\right)^{1/2}}\right] \tag{120}$$

$$A = \arctan\left[\frac{x_T}{y_T}\right] \tag{121}$$

and the partial derivatives with respect to the six orbital elements in the form of position and velocity coordinates are

$$\begin{bmatrix} \frac{\partial E}{\partial x} \\ \frac{\partial E}{\partial y} \\ \\ \frac{\partial E}{\partial z} \end{bmatrix} = \frac{z_T}{r_T^2\left(r_T^2 - z_T^2\right)^{1/2}} \quad [S]^T \begin{bmatrix} -x_T \\ -y_T \\ \\ \frac{r_T^2 - z_T^2}{z_T} \end{bmatrix} \tag{122}$$

$$\frac{\partial E}{\partial \dot{x}} = \frac{\partial E}{\partial \dot{y}} = \frac{\partial E}{\partial \dot{z}} = 0 \tag{123}$$

and

$$\begin{bmatrix} \frac{\partial A}{\partial x} \\ \frac{\partial A}{\partial y} \\ \frac{\partial A}{\partial z} \end{bmatrix} = \frac{1}{r_T^2 - z_T^2} \begin{bmatrix} s_{11} & s_{21} \\ s_{12} & s_{22} \\ s_{13} & s_{23} \end{bmatrix} \begin{bmatrix} y_T \\ -x_T \end{bmatrix} \tag{124}$$

$$\frac{\partial A}{\partial \dot{x}} = \frac{\partial A}{\partial \dot{y}} = \frac{\partial A}{\partial \dot{z}} = 0 \tag{125}$$

where

$$r_T = \left(x_T^2 + y_T^2 + z_T^2\right)^{1/2}$$

and the matrix S has been defined previously in Section III-F.

The expressions for the estimated right ascension, α, and declination, δ, must be obtained in the particular celestial system in which the observations were obtained. It must be noted that, in practice, the observed right ascension and declination are obtained in an astrometric system, which is defined by the coordinates given to a number of stars. Thus the difference between a celestial and an astrometric system will be due to the errors in the astrometric system. With this understanding, we will designate the celestial system as the reference system.

After transformation of the satellite and station coordinates into the required celestial system (see Section III-F), the estimated right ascension and declination can be obtained from the following relationships.

$$\alpha = \arctan\left[\frac{y_0 - y_{s_0}}{x_0 - x_{s_0}}\right] \tag{126}$$

$$\delta = \arctan\left[\frac{z_0 - z_{s_0}}{r_{x_0}}\right] \tag{127}$$

where the subscript, 0, indicates satellite coordinates in the celestial system and subscript s_0 indicates station coordinates in the celestial system.

$$r_{X_0} = \left[\left(x_0 - x_{s_0}\right)^2 + \left(y_0 - y_{s_0}\right)^2\right]^{1/2} \tag{128}$$

The partial derivatives are obtained from the above equations giving

$$\begin{bmatrix} \frac{\partial \alpha}{\partial x} \\ \frac{\partial \alpha}{\partial y} \end{bmatrix} = \frac{1}{r_{X_0}^2} \begin{bmatrix} \ell_{11} & \ell_{21} \\ \ell_{12} & \ell_{22} \end{bmatrix} \begin{bmatrix} -\left(y_0 - y_{s_0}\right) \\ \left(x_0 - x_{s_0}\right) \end{bmatrix} \tag{129}$$

and

$$\frac{\partial \alpha}{\partial z} = \frac{\partial \alpha}{\partial \dot{x}} = \frac{\partial \alpha}{\partial \dot{y}} = \frac{\partial \alpha}{\partial \dot{z}} = 0 \tag{130}$$

$$\begin{bmatrix} \frac{\partial \delta}{\partial x} \\ \frac{\partial \delta}{\partial y} \\ \\ \frac{\partial \delta}{\partial z} \end{bmatrix} = \frac{1}{r_T^2} [L]^T \begin{bmatrix} -\frac{1}{r_{X_0}} \left(z_0 - z_{s_0}\right)\left(x_0 - x_{s_0}\right) \\ -\frac{1}{r_{X_0}} \left(z - z_{s_0}\right)\left(y_0 - y_{s_0}\right) \\ \\ r_{X_0} \end{bmatrix} \tag{131}$$

$$\frac{\partial \delta}{\partial \dot{x}} = \frac{\partial \delta}{\partial \dot{y}} = \frac{\partial \delta}{\partial \dot{z}} = 0 \tag{132}$$

where

$$[L] = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} = P_{ST}\ P^T\ N^T$$

is the transformation matrix from the true equinox and equator of date to the mean equinox and equator of the particular celestial system (see Section III-F). The radius

$$r_T = \left[r_{X_0}^2 \quad \left(z_0 - z_{s0} \right)^2 \right]^{1/2} \tag{133}$$

The matrix L normally will be close to a unit matrix, and in many cases, the multiplication of the partial derivatives by this matrix may not be necessary.

The elevation and azimuth rates can be obtained by differentiation of the expressions for E and A with respect to time.

$$\dot{E} = \frac{\dot{z}_T \, r_{XT} - z_T \, \dot{r}_{XT}}{r_T^2} \tag{134}$$

$$\dot{A} = \frac{\dot{x}_T y_T - \dot{y}_T x_T}{x_T^2 + y_T^2} \tag{135}$$

and the partial derivatives of the elevation rate with respect to the orbital elements are

$$\begin{bmatrix} \frac{\partial \dot{E}}{\partial x} \\ \frac{\partial \dot{E}}{\partial y} \\ \frac{\partial \dot{E}}{\partial z} \end{bmatrix} = \frac{1}{r_T^2} \left[S \right]^T \begin{bmatrix} \left[\dot{z}_T x_T - \frac{z_T}{r_{XT}} \left(\dot{x}_T \, r_{XT} - \dot{r}_{XT} \, x_T \right) \right] \frac{1}{r_{XT}} - 2x_T \, \dot{E} \\ \left[\dot{z}_T y_T - \frac{z_T}{r_{XT}} \left(\dot{y}_T \, r_{XT} - \dot{r}_{XT} \, y_T \right) \right] \frac{1}{r_{XT}} - 2y_T \, \dot{E} \\ - \left(\dot{r}_{XT} + 2z_T \, \dot{E} \right) \end{bmatrix} \tag{136}$$

$$\begin{bmatrix} \frac{\partial \dot{E}}{\partial \dot{x}} \\ \frac{\partial \dot{E}}{\partial \dot{y}} \\ \frac{\partial \dot{E}}{\partial \dot{z}} \end{bmatrix} = \frac{1}{r_T^2} \left[S \right]^T \begin{bmatrix} - \frac{x_T z_T}{r_{XT}} \\ - \frac{y_T z_T}{r_{XT}} \\ r_{XT} \end{bmatrix} \tag{137}$$

Similarly, the partial derivatives of the azimuth rate with respect to the orbital elements are

$$\begin{bmatrix} \frac{\partial \dot{A}}{\partial x} \\ \\ \frac{\partial \dot{A}}{\partial y} \\ \\ \frac{\partial \dot{A}}{\partial z} \end{bmatrix} = \frac{1}{x_T^2 + y_T^2} [S]^T \begin{bmatrix} -(\dot{y}_T + 2\dot{A}\, x_T) \\ \\ (\dot{x}_T - 2\dot{A}\, y_T) \\ \\ 0 \end{bmatrix} \tag{138}$$

$$\begin{bmatrix} \frac{\partial \dot{A}}{\partial \dot{x}} \\ \\ \frac{\partial \dot{A}}{\partial \dot{y}} \\ \\ \frac{\partial \dot{A}}{\partial \dot{z}} \end{bmatrix} = \frac{1}{x_T^2 + y_T^2} [S]^T \begin{bmatrix} y_T \\ \\ -x_T \\ \\ 0 \end{bmatrix} \tag{139}$$

where

$$r_T = \left(x_T^2 + y_T^2 + z_T^2\right)^{1/2}$$

$$r_{XT} = \left(x_T^2 + y_T^2\right)^{1/2}$$

$$\dot{r}_{XT} = \frac{x_T \dot{x}_T + y_T \dot{y}_T}{r_{XT}}$$

and the matrix S has been defined previously (see Section III-F).

Next we will consider the expressions for range, range rate, and range acceleration, which are, respectively

$$r_T = \left(x_T^2 + y_T^2 + z_T^2\right)^{1/2} \tag{140}$$

$$\dot{r}_T = \frac{1}{r_T}\left(x_T \dot{x}_T + y_T \dot{y}_T + z_T \dot{z}_T\right)$$

$$= \frac{1}{r_T}\left[(x - x_s)(\dot{x} + \omega_E y_s) + (y - y_s)(\dot{y} - \omega_E x_s) + (z - z_s)\dot{z}\right] \tag{141}$$

$$\ddot{r}_T = \frac{1}{r_T}\left[\left(\dot{x}^2 - \omega_E^2 y_s^2\right) + (x - x_s)\left(\ddot{x} - \omega_E^2 x_s\right) + \right.$$

$$+ (\dot{y}^2 - \omega_E^2 x_s^2) + (y - y_s)(\ddot{y} - \omega_E^2 y_s)$$

$$+ \dot{z}^2 + (z - z_s)\ddot{z} - \dot{r}_T^2 \Big] \tag{142}$$

Where ω_E is the rate of Earth's rotation, $\omega_E = 0.000072921150$ radians per mean solar second. The subscript s indicates station coordinates.

The partial derivatives of range with respect to the orbital elements are

$$\begin{bmatrix} \frac{\partial r_T}{\partial x} \\ \frac{\partial r_T}{\partial y} \\ \frac{\partial r_T}{\partial z} \end{bmatrix} = \frac{1}{r_T} \begin{bmatrix} x - x_s \\ y - y_s \\ z - z_s \end{bmatrix} \tag{143}$$

$$\frac{\partial r_T}{\partial \dot{x}} = \frac{\partial r_T}{\partial \dot{y}} = \frac{\partial r_T}{\partial \dot{z}} = 0 \tag{144}$$

and the corresponding derivatives of the range rate

$$\begin{bmatrix} \frac{\partial \dot{r}_T}{\partial x} \\ \frac{\partial \dot{r}_T}{\partial y} \\ \frac{\partial \dot{r}_T}{\partial z} \end{bmatrix} = \frac{1}{r_T} \begin{bmatrix} (\dot{x} + \omega_E y_s) - \frac{\dot{r}_T}{r_T}(x - x_s) \\ (\dot{y} - \omega_E x_s) - \frac{\dot{r}_T}{r_T}(y - y_s) \\ \dot{z} - \frac{\dot{r}_T}{r_T}(z - z_s) \end{bmatrix} \tag{145}$$

$$\begin{bmatrix} \frac{\partial \dot{r}_T}{\partial \dot{x}} \\ \frac{\partial \dot{r}_T}{\partial \dot{y}} \\ \frac{\partial \dot{r}_T}{\partial \dot{z}} \end{bmatrix} = \frac{1}{r_T} \begin{bmatrix} x - x_s \\ y - y_s \\ z - z_s \end{bmatrix} \tag{146}$$

Finally, the derivatives of the range acceleration are:

$$\begin{bmatrix} \frac{\partial \ddot{r}_T}{\partial x} \\ \frac{\partial \ddot{r}_T}{\partial y} \\ \frac{\partial \ddot{r}_T}{\partial z} \end{bmatrix} = \frac{1}{r_T^2} \begin{bmatrix} r_T \left[(\ddot{x} - \omega_E^2 x_s) - 2\dot{r}_T \frac{\partial \dot{r}_T}{\partial x} \right] - \ddot{r}_T (x - x_s) \\ r_T \left[(\ddot{y} - \omega_E^2 y_s) - 2\dot{r}_T \frac{\partial \dot{r}_T}{\partial y} \right] - \ddot{r}_T (y - y_s) \\ r_T \left(\ddot{z} - 2\dot{r}_T \frac{\partial \dot{r}_T}{\partial z} \right) - \ddot{r}_T (z - z_s) \end{bmatrix} \tag{147}$$

$$\begin{bmatrix} \frac{\partial \ddot{r}_T}{\partial \dot{x}} \\ \frac{\partial \ddot{r}_T}{\partial \dot{y}} \\ \frac{\partial \ddot{r}_T}{\partial \dot{z}} \end{bmatrix} = \frac{2}{r_T} \begin{bmatrix} \dot{x} - \frac{\dot{r}_T}{r_T} (x - x_s) \\ \dot{y} - \frac{\dot{r}_T}{r_T} (y - y_s) \\ \dot{z} - \frac{\dot{r}_T}{r_T} (z - z_s) \end{bmatrix} \tag{148}$$

VIII. EQUATIONS OF MOTION

A. ASPECTS OF THE MATHEMATICAL MODEL

The mathematical model of the dynamical system is expressed by the equations of motion with terms representing the various forces acting on the satellite. The main forces acting on a close Earth satellite are due to Earth's gravitational field, atmospheric drag, Sun's and Moon's gravitational fields and solar radiation pressure. The effect of these forces can vary considerably, depending on the particular orbit. For example, a circular orbit of an Earth satellite at 800 km altitude will be perturbed by a maximum of about 1.5 km due to the second harmonic in the Earth's gravitational potential, while the maximum perturbation due to the combined gravitational attraction of Moon and Sun will be less than a meter. The perturbations due to atmospheric drag and solar radiation pressure may be of the same order of magnitude at this altitude, while at low altitudes, the effect of drag will provide the principal force. Therefore, for a close Earth satellite, the perturbations due to the gravitational attraction of the Moon and the Sun, and the perturbations due to the solar radiation pressure may be computed considering only first-order effects.

The equations of motion for a close satellite are written in the previously defined basic axes system.

$$\ddot{x} = X_G + X_D + X_\odot + X_\leftmoon + X_{SR} \tag{149a}$$

$$\ddot{y} = Y_G + Y_D + Y_\odot + Y_\leftmoon + Y_{SR} \tag{149b}$$

$$\ddot{z} = Z_G + Z_D + Z_\odot + Z_\leftmoon + Z_{SR} \tag{149c}$$

The terms with subscripts G, D, $\odot$, $\leftmoon$ and SR represent the components of acceleration due to Earth's gravitational field, atmospheric drag, Sun's attraction, Moon's attraction and solar radiation pressure, respectively. For accurate computations, the independent time argument in the equations of motion must be in a uniform time scale, such as the atomic time (see Chapter II).

We will now develop the terms in a form convenient for integration in rectangular coordinates.

B. EARTH'S GRAVITATIONAL FIELD

The mathematical representation of the Earth's gravitational field is expressed by means of the gravitational potential function which can be written

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R_E}{r}\right)^n P_n^m(\sin\phi)\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right] \tag{150}$$

$$C_{no} = -J_n = C_n,\quad S_{no} = 0$$

where $\mu = G M_{\oplus}$

J_n, C_{nm}, and S_{nm} are numerical coefficients, R_E the mean equatorial radius of the Earth, r the distance of the satellite from the center of the Earth, ϕ the latitude, and P_n^m the associated Legendre polynomial

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m} \tag{151}$$

where P_n is the Legendre polynominal. The longitude λ is to be counted positive to the east in this application.

The harmonics represented in the gravitational potential function are called spherical harmonics (Ref. 18). If $0 < m < n$ they are called tesseral harmonics as a special case of the spherical harmonics. If $m = 0$, they are called zonal harmonics, and if $m = n$, they are called sectorial harmonics. The gravitational potential for bodies with spherical symmetry can be expressed by the zonal harmonics only, i.e., the potential is a function of latiiude and independent of longitude. For bodies of arbitrary shape, the potential must include the tesseral harmonics, which are dependent on both latitude and longitude.

The Legendre polynomials are computed from the general expression:

$$P_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 2\cdot 3\cdots n}\left\{\left(\frac{z}{r}\right)^n - \frac{n(n-1)}{2(2n-1)}\left(\frac{z}{r}\right)^{n-2}\right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}\left(\frac{z}{r}\right)^{n-4} - \cdots\right\} \tag{152}$$

The components of acceleration due to the gravitational potential are

$$X_G = \frac{\partial U}{\partial x} \tag{153a}$$

$$Y_G = \frac{\partial U}{\partial y} \tag{153b}$$

$$Z_G = \frac{\partial U}{\partial z} \tag{153c}$$

Before we differentiate the potential function, we will write U in the following form

$$U = \frac{\mu}{r} + U_Z + U_T \tag{154}$$

where μ/r represents the potential of the total mass concentrated at the center of the body. The potential functions representing the zonal and tesseral harmonics are U_Z and U_T, respectively. By differentiating μ/r

$$\frac{\partial}{\partial x}\left(\frac{\mu}{r}\right) = -\frac{\mu x}{r^3} \tag{155a}$$

$$\frac{\partial}{\partial y}\left(\frac{\mu}{r}\right) = -\frac{\mu y}{r^3} \tag{155b}$$

$$\frac{\partial}{\partial z}\left(\frac{\mu}{r}\right) = -\frac{\mu z}{r^3} \tag{155c}$$

where

$$r^2 = x^2 + y^2 + z^2$$

and

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now, computing the Legendre polynomials and substituting into the potential function, we obtain for the first nine zonal harmonics

$$\begin{aligned} U_Z = -\frac{\mu}{r}\Bigg\{ & \frac{J_2}{2}\left(\frac{R_E}{r}\right)^2 \left[3\left(\frac{z}{r}\right)^2 - 1\right] \\ & + \frac{J_3}{2}\left(\frac{R_E}{r}\right)^3 \left[5\left(\frac{z}{r}\right)^3 - 3\frac{z}{r}\right] \\ & + \frac{J_4}{8}\left(\frac{R_E}{r}\right)^4 \left[35\left(\frac{z}{r}\right)^4 - 30\left(\frac{z}{r}\right)^2 + 3\right] \\ & + \frac{J_5}{8}\left(\frac{R_E}{r}\right)^5 \left[63\left(\frac{z}{r}\right)^5 - 70\left(\frac{z}{r}\right)^3 + 15\frac{z}{r}\right] \end{aligned}$$

$$+\frac{J_6}{16}\left(\frac{R_E}{r}\right)^6\left[231\left(\frac{z}{r}\right)^6-315\left(\frac{z}{r}\right)^4+105\left(\frac{z}{r}\right)^2-5\right]$$

$$+\frac{J_7}{16}\left(\frac{R_E}{r}\right)^7\left[429\left(\frac{z}{r}\right)^7-693\left(\frac{z}{r}\right)^5+315\left(\frac{z}{r}\right)^3-35\frac{z}{r}\right]$$

$$+\frac{J_8}{128}\left(\frac{R_E}{r}\right)^8\left[6435\left(\frac{z}{r}\right)^8-12,012\left(\frac{z}{r}\right)^6+6930\left(\frac{z}{r}\right)^4-1260\left(\frac{z}{r}\right)^2+35\right]$$

$$+\frac{J_9}{384}\left(\frac{R_E}{r}\right)^9\left[36,465\left(\frac{z}{r}\right)^9-77,220\left(\frac{z}{r}\right)^7+54,054\left(\frac{z}{r}\right)^5-13,860\left(\frac{z}{r}\right)^3+945\frac{z}{r}\right]\Bigg\} \tag{156}$$

The partial differentials of the above function with respect to x, y, z can be written in the form

$$\frac{\partial U_z}{\partial x}=\left(\frac{\partial U_z}{\partial r}\right)_z\frac{\partial r}{\partial x} \tag{157a}$$

$$\frac{\partial U_z}{\partial y}=\left(\frac{\partial U_z}{\partial r}\right)_z\frac{\partial r}{\partial y} \tag{157b}$$

$$\frac{\partial U_z}{\partial z}=\left(\frac{\partial U_z}{\partial r}\right)_z\frac{\partial r}{\partial z}+\left(\frac{\partial U_z}{\partial z}\right)_r \tag{157c}$$

where $\left(\frac{\partial U_z}{\partial r}\right)_z$ designates the partial derivative of U_z with respect to r when z is kept constant, and $\left(\frac{\partial U_z}{\partial z}\right)_r$ the derivative with respect to z when r is kept constant.

By performing the differentiations we obtain

$$\left(\frac{\partial U_z}{\partial r}\right)_z=-\frac{\mu}{r^2}\Bigg\{\left(\frac{R_E}{r}\right)^2\left[\left(-\frac{15}{2}\right)\left(\frac{z}{r}\right)^2+\frac{3}{2}\right]J_2$$

$$+\left(\frac{R_E}{r}\right)^3\left[\left(-\frac{35}{2}\right)\left(\frac{z}{r}\right)^3+\left(\frac{15}{2}\right)\left(\frac{z}{r}\right)\right]J_3$$

$$+\left(\frac{R_E}{r}\right)^4\left[\left(-\frac{315}{8}\right)\left(\frac{z}{r}\right)^4+\left(\frac{210}{8}\right)\left(\frac{z}{r}\right)^2+\left(-\frac{15}{8}\right)\right]J_4+\cdots\Bigg\}$$

and

$$\left(\frac{\partial U_z}{\partial z}\right)_r = \frac{\mu}{r^2}\left\{\left(\frac{R_E}{r}\right)^2 (-3)\left(\frac{z}{r}\right) J_2 + \left(\frac{R_E}{r}\right)^3 \left[\left(-\frac{15}{2}\right)\left(\frac{z}{r}\right)^2 + \frac{3}{2}\right] J_3 + \left(\frac{R_E}{r}\right)^4 \left[\left(-\frac{35}{2}\right)\left(\frac{z}{r}\right)^3 + \left(\frac{15}{2}\right)\left(\frac{z}{r}\right)\right] J_4 + \cdots\right\}$$

Comparing the two derivatives and designating

$$f_1 = -3\frac{z}{r}$$

$$f_2 = -7.5\left(\frac{z}{r}\right)^2 + 1.5$$

$$f_3 = -17.5\left(\frac{z}{r}\right)^3 + 7.5\frac{z}{r}$$

$$f_4 = -39.375\left(\frac{z}{r}\right)^4 + 26.25\left(\frac{z}{r}\right)^2 - 1.875$$

$$f_5 = -86.625\left(\frac{z}{r}\right)^5 + 78.75\left(\frac{z}{r}\right)^3 - 13.125\frac{z}{r}$$

$$f_6 = -187.6875\left(\frac{z}{r}\right)^6 + 216.5625\left(\frac{z}{r}\right)^4 - 59.0625\left(\frac{z}{r}\right)^2 + 2.1875$$

$$f_7 = -402.1875\left(\frac{z}{r}\right)^7 + 563.0625\left(\frac{z}{r}\right)^5 - 216.5625\left(\frac{z}{r}\right)^3 + 19.6875\frac{z}{r}$$

$$f_8 = -854.648437\left(\frac{z}{r}\right)^8 + 1407.65625\left(\frac{z}{r}\right)^6 - 703.828125\left(\frac{z}{r}\right)^4 + 108.28125\left(\frac{z}{r}\right)^2 - 2.4609375$$

$$f_9 = -1804.25781\left(\frac{z}{r}\right)^9 + 3418.59375\left(\frac{z}{r}\right)^7 - 2111.48437\left(\frac{z}{r}\right)^5 + 469.21875\left(\frac{z}{r}\right)^3 - 27.0703125\frac{z}{r}$$

we can write

$$\left(\frac{\partial U_z}{\partial r}\right)_z = -\frac{\mu}{r^2}\left[\left(\frac{R_E}{r}\right)^2 f_2 J_2 + \left(\frac{R_E}{r}\right)^3 f_3 J_3 + \cdots + \left(\frac{R_E}{r}\right)^9 f_9 J_9\right] \qquad (158)$$

$$\left(\frac{\partial U_Z}{\partial z}\right)_r = \frac{\mu}{r^2}\left[\left(\frac{R_E}{r}\right)^2 f_1 J_2 + \left(\frac{R_E}{r}\right)^3 f_2 J_3 + \cdots + \left(\frac{R_E}{r}\right)^9 f_8 J_9\right] \quad (159)$$

and, designating the two functions in the brackets as F_1 and F_2, respectively, we can express the accclerations due to the zonal harmonics as

$$\frac{\partial U_Z}{\partial x} = X_Z = -\frac{\mu x}{r^3} F_1 \quad (160a)$$

$$\frac{\partial U_Z}{\partial y} = Y_Z = -\frac{\mu y}{r^3} F_1 \quad (160b)$$

$$\frac{\partial U_Z}{\partial z} = Z_Z = -\frac{\mu}{r^2}\left(\frac{z}{r} F_1 - F_2\right) \quad (160c)$$

We will now develop the tesseral harmonics considering the gravitational potential up to the fourth order harmonics.

$$U_T = \frac{\mu}{r}\left[\sum_{n=2}^{n=4}\sum_{m=1}^{n}\left(\frac{R_E}{r}\right)^n P_n^m (\sin\phi)\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right] \quad (161)$$

For integration in rectangular coordinates, it is convenient to express the trigonometric functions in terms of x", y", z", which are the satellite coordinates in an Earth fixed terrestrial sysiem. The integration itself is done in the basic system defined in Section III-B. The required coordinate transformations are given in Section III. To facilitate writing we will adapt the notation

$$x \equiv x''$$

$$y \equiv y''$$

$$z \equiv z''$$

With this notation, the trigonometric functions can be expressed as

$$\sin\phi = \frac{z}{r}$$

$$\cos\phi = \frac{\left(x^2 + y^2\right)^{1/2}}{r}$$

where

$$r^2 = x^2 + y^2 + z^2$$

The associated Legendre polynomials then are

$$P_2^2 = 3\,\frac{x^2 + y^2}{r^2}$$

$$P_3^1 = \frac{3}{2}\,\frac{\left(5z^2 - r^2\right)\left(x^2 + y^2\right)^{1/2}}{r^3}$$

$$P_3^2 = 15\,\frac{z\,(x^2 + y^2)}{r^3}$$

$$P_3^3 = 15\,\frac{\left(x^2 + y^2\right)^{3/2}}{r^3}$$

$$P_4^1 = \frac{5}{2}\,\frac{z\left(7z^2 - 3r^2\right)\left(x^2 + y^2\right)^{1/2}}{r^4}$$

$$P_4^2 = \frac{15}{2}\,\frac{(7z^2 - r^2)\,(x^2 + y^2)}{r^4}$$

$$P_4^3 = 105\,\frac{z\left(x^2 + y^2\right)^{3/2}}{r^4}$$

$$P_4^4 = 105\,\frac{\left(x^2 + y^2\right)^2}{r^4}$$

Similarly, we can express the longitude dependent functions in terms of the rectangular coordinates.

Then we can write

$$U_T = \mu(f_{22} + f_{31} + f_{32} + f_{33} + f_{41} + f_{42} + f_{43} + f_{44}) \qquad (162)$$

where

$$f_{22} = \left(\frac{R_E}{r}\right)^2 \frac{3}{r^3}\left[C_{22}\ (x^2 - y^2) + S_{22}\ (2xy)\right]$$

$$f_{31} = \left(\frac{R_E}{r}\right)^3 \frac{3\ (5z^2 - r^2)}{2r^4}\left[C_{31}\ x + S_{31}\ y\right]$$

$$f_{32} = \left(\frac{R_E}{r}\right)^3 \frac{15z}{r^4}\left[C_{32}\ (x^2 - y^2) + S_{32}\ (2xy)\right]$$

$$f_{33} = \left(\frac{R_E}{r}\right)^3 \frac{15}{r^4}\left[C_{33}\ x\ (x^2 - 3y^2) + S_{33}\ y\ (3x^2 - y^2)\right]$$

$$f_{41} = \left(\frac{R_E}{r}\right)^4 \frac{5z\ (7z^2 - 3r^2)}{2r^5}\left[C_{41}\ x + S_{41}\ y\right]$$

$$f_{42} = \left(\frac{R_E}{r}\right)^4 \frac{15\ (7z^2 - r^2)}{2r^5}\left[C_{42}\ (x^2 - y^2) + S_{42}\ (2xy)\right]$$

$$f_{43} = \left(\frac{R_E}{r}\right)^4 \frac{105z}{r^5}\left[C_{43}\ x\ (x^2 - 3y^2) + S_{43}\ y\ (3x^2 - y^2)\right]$$

$$f_{44} = \left(\frac{R_E}{r}\right)^4 \frac{105}{r^5}\left[C_{44}\ (x^4 - 6x^2y^2 + y^4) + S_{44}\ 4xy\ (x^2 - y^2)\right]$$

Performing the differentiations we obtain

$$X_T = \frac{\partial U_T}{\partial x} = \frac{3\mu R_E^2}{r^7}\left\{C_{22}\ x\left[2r^2 - 5\ (x^2 - y^2)\right] + S_{22}\ 2y\ (r^2 - 5x^2)\right\}$$

$$+ \frac{3\mu R_E^3}{2r^9}\left\{C_{31}\left[5x^2\ (r^2 - 7z^2) + (5z^2 - r^2)\ r^2\right] + S_{31}\ 5xy\ (r^2 - 7z^2)\right\}$$

$$+ \frac{15\mu R_E^3 z}{r^9}\left\{C_{32}\ x\left[2r^2 - 7\ (x^2 - y^2)\right] + S_{32}\ 2y\ (r^2 - 7x^2)\right\}$$

$$+\frac{15\mu R_E^3}{r^9}\left\{C_{33}\left[3r^2\ (x^2-y^2)-7x^2\ (x^2-3y^2)\right]+S_{33}\ xy\left[6r^2-7(3x^2-y^2)\right]\right\}$$

$$+\frac{5\mu R_E^4 z}{2r^{11}}\left\{C_{41}\left[7z^2\ (r^2-9x^2)-3r^2\ (r^2-7x^2)\right]+S_{41}\ 21xy\ (r^2-3z^2)\right\}$$

$$+\frac{15\mu R_E^4}{2r^{11}}\left\{C_{42}\ x\left[7\ (r^2-9z^2)\ (x^2-y^2)+2r^2\ (7z^2-r^2)\right]\right.$$

$$\left.+S_{42}\ 2y\left[7x^2\ (r^2-9z^2)+r^2\ (7z^2-r^2)\right]\right\}$$

$$+\frac{315\mu R_E^4 z}{r^{11}}\left\{C_{43}\left[r^2\ (x^2-y^2)-3x^2\ (x^2-3y^2)\right]+S_{43}\ xy\left[2r^2-3\ (3x^2-y^2)\right]\right\}$$

$$+\frac{105\mu R_E^4}{r^{11}}\left\{C_{44}\ x\left[4r^2\ (x^2-3y^2)-9\ (x^4-6x^2y^2+y^4)\right]\right.$$

$$\left.+S_{44}\ 4y\left[r^2\ (3x^2-y^2)-9x^2\ (x^2-y^2)\right]\right\} \qquad (163)$$

$$Y_T=\frac{\partial U_T}{\partial y}=\frac{3\mu R_E^2}{r^7}\left\{C_{22}\ y\left[-2r^2-5\ (x^2-y^2)\right]+S_{22}\ 2x\ (r^2-5y^2)\right\}$$

$$+\frac{3\mu R_E^3}{2r^9}\left\{C_{31}\left[5xy\ (r^2-7z^2)\right]+S_{31}\left[5y^2\ (r^2-7z^2)+r^2\ (5z^2-r^2)\right]\right\}$$

$$+\frac{15\mu R_E^3 z}{r^9}\left\{C_{32}\ y\left[-2r^2-7\ (x^2-y^2)\right]+S_{32}\ 2x\ (r^2-7y^2)\right\}$$

$$+\frac{15\mu R_E^3}{r^9}\left\{C_{33}\ xy\left[-6r^2-7\ (x^2-3y^2)\right]+S_{33}\left[3r^2\ (x^2-y^2)-7y^2\ (3x^2-y^2)\right]\right\}$$

$$+\frac{5\mu R_E^4 z}{2r^{11}}\left\{C_{41}\ 21xy\ (r^2-3z^2)+S_{41}\left[7z^2\ (r^2-9y^2)+3r^2\ (7y^2-r^2)\right]\right\}$$

$$+\frac{15\mu R_E^4}{2r^{11}}\left\{C_{42}\ y\left[7\ (r^2-9z^2)\ (x^2-y^2)-2r^2\ (7z^2-r^2)\right]\right.$$

$$\left.+S_{42}\ 2x\left[7y^2\ (r^2-9z^2)+r^2\ (7z^2-r^2)\right]\right\}$$

$$+ \frac{315\mu R_E^4 z}{r^{11}} \left\{ C_{43} \, xy \left[-2r^2 - 3 (x^2 - 3y^2) \right] + S_{43} \left[r^2 (x^2 - y^2) - 3y^2 (3x^2 - y^2) \right] \right\}$$

$$+ \frac{105\mu R_E^4}{r^{11}} \left\{ C_{44} \, y \left[4r^2 (y^2 - 3x^2) - 9 (x^4 - 6x^2 y^2 + y^4) \right] \right.$$

$$\left. + S_{44} \, 4x \left[r^2 (x^2 - 3y^2) - 9y^2 (x^2 - y^2) \right] \right\} \qquad (164)$$

$$Z_T = \frac{\partial U_T}{\partial z} = - \frac{15\mu R_E^2 z}{r^7} \left[C_{22} \, (x^2 - y^2) + S_{22} \, (2xy) \right]$$

$$+ \frac{15\mu R_E^3 z}{2r^9} \, (3r^2 - 7z^2) \left[C_{31} \, x + S_{31} y \right]$$

$$+ \frac{15\mu R_E^3}{r^9} \, (r^2 - 7z^2) \left[C_{32} \, (x^2 - y^2) + S_{32} \, (2xy) \right]$$

$$- \frac{105\mu R_E^3 z}{r^9} \left[C_{33} \, x \, (x^2 - 3y^2) + S_{33} \, y \, (3x^2 - y^2) \right]$$

$$+ \frac{15\mu R_E^4}{2r^{11}} \, (14z^2 r^2 - r^4 - 21z^4) \left[C_{41} \, x + S_{41} \, y \right]$$

$$+ \frac{315\mu R_E^4 z}{2r^{11}} \, (r^2 - 3z^2) \left[C_{42} \, (x^2 - y^2) + S_{42} \, (2xy) \right]$$

$$+ \frac{105\mu R_E^4}{r^{11}} \, (r^2 - 9z^2) \left[C_{43} \, x \, (x^2 - 3y^2) + S_{43} \, y \, (3x^2 - y^2) \right]$$

$$- \frac{945\mu R_E^4 z}{r^{11}} \left[C_{44} \, (x^4 - 6x^2 y^2 + y^4) + S_{44} \, 4xy \, (x^2 - y^2) \right] \qquad (165)$$

The accelerations due to the gravitational potential then can be written

$$X_G = - \frac{\mu x}{r^3} + X_Z + X_T \qquad (166a)$$

$$Y_G = -\frac{\mu Y}{r^3} + Y_Z + Y_T \tag{166b}$$

$$Z_G = -\frac{\mu Z}{r^3} + Z_Z + Z_T \tag{166c}$$

C. ATMOSPHERIC DRAG

The acceleration due to drag on a satellite is a function of atmospheric density, ρ, relative velocity of the satellite with respect to the atmosphere, v_r, satellite mass, m, drag coefficient, C_D, and reference area, A.

The atmospheric density can vary considerably and is rather difficult to evaluate with an accuracy necessary for precise orbit determination, particularly at lower altitudes where its effect is large. It is a function of altitude, exospheric temperature (e.g. Ref. 19), and the relative position of the Sun.

If the density is given, the expression for the acceleration is

$$a_D = \frac{\rho v_r^2}{2} \frac{C_D A}{m} \tag{167}$$

The relative velocity, v_r, can be obtained on the assumption that the atmosphere rotates with the same angular velocity as the Earth. The relative velocity components in the basic axes system then are

$$v_x = \dot{x} + \omega_E y \tag{168a}$$

$$v_y = \dot{y} - \omega_E x \tag{168b}$$

$$v_z = \dot{z} \tag{168c}$$

Where ω_E is the angular velocity of the Earth's rotation.

Then

$$v_r = (v_x^2 + v_y^2 + v_z^2)^{1/2} \tag{169}$$

and the components of the accelerations in the basic axes system are

$$X_D = -\rho v_r v_x \frac{C_D A}{2m} \tag{170a}$$

$$Y_D = -\rho v_r v_y \frac{C_D A}{2m} \tag{170b}$$

$$Z_D = -\rho v_r v_z \frac{C_D A}{2m} \tag{170c}$$

D. SUN'S GRAVITATIONAL ATTRACTION

The perturbing action of the Sun on a elose satellite orbit can be expressed by the so called disturbing function. If the ratio of the satellite distance from the center of Earth to the Sun's distanee is sufficiently small, the disturbing function can be expanded in power series whieh converge rapidly. The disturbing function is given in a general form as

$$R = \mu_\odot \left(\frac{1}{\Delta} - \frac{x x_\odot + y y_\odot + z z_\odot}{r_\odot^3} \right) \tag{171}$$

where

$$\mu_\odot = G M_\odot$$

and x, y, z and $x_\odot$, $y_\odot$, $z_\odot$ are the satellite and Sun's coordinates, respeetively. $\Delta$is the distance between the satellite and the Sun,

$$\Delta^2 = (x - x_\odot)^2 + (y - y_\odot)^2 + (z - z_\odot)^2 \tag{172}$$

The disturbing function can be expanded in powers of $r/r_\odot$ (Ref. 20) giving

$$R = \frac{\mu_\odot}{r_\odot} \left[\left(\frac{r}{r_\odot} \right)^2 \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) + \left(\frac{r}{r_\odot} \right)^3 \left(\frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right) + \ldots \right] \tag{173}$$

where the functions in the brackets are the Legendre polynomials which were treated in Chapter VIII, Section B.

θ is the angle between the direetions to the satellite and the Sun.

Consequently,

$$\cos \theta = \frac{x x_\odot + y y_\odot + z z_\odot}{r r_\odot} \tag{174}$$

and we can write the disturbing function neglecting all higher order terms as

$$R_{\odot} = \frac{\mu_{\odot}}{r_{\odot}^3}\left[\frac{3}{2r_{\odot}^2}\left(x x_{\odot} + y y_{\odot} + z z_{\odot}\right)^2 - \frac{1}{2}\left(x^2 + y^2 + z^2\right)\right] \tag{175}$$

The accelerations due to the Sun's attraction then are obtained by differentiation.

$$X_{\odot} = \frac{\partial R_{\odot}}{\partial x} = \frac{\mu_{\odot}}{r_{\odot}^3}\left[3r\,\xi_{\odot}\left(\xi\xi_{\odot} + \eta\eta_{\odot} + \zeta\zeta_{\odot}\right) - x\right] \tag{176a}$$

$$Y_{\odot} = \frac{\partial R_{\odot}}{\partial y} = \frac{\mu_{\odot}}{r_{\odot}^3}\left[3r\,\eta_{\odot}\left(\xi\xi_{\odot} + \eta\eta_{\odot} + \zeta\zeta_{\odot}\right) - y\right] \tag{176b}$$

$$Z_{\odot} = \frac{\partial R_{\odot}}{\partial z} = \frac{\mu_{\odot}}{r_{\odot}^3}\left[3r\,\zeta_{\odot}\left(\xi\xi_{\odot} + \eta\eta_{\odot} + \zeta\zeta_{\odot}\right) - z\right] \tag{176c}$$

where ξ, η, ζ, and $\xi_{\odot}$, $\eta_{\odot}$, $\zeta_{\odot}$ are the direction cosines of the satellite and the Sun's radius vector, respectively.

The direction cosines of the Sun can be obtained with sufficient accuracy from the following equations.

$$\xi_{\odot} = \cos \nu_{\odot} \tag{177a}$$

$$\eta_{\odot} = \sin \nu_{\odot} \cos \epsilon \tag{177b}$$

$$\zeta_{\odot} = \sin \nu_{\odot} \sin \epsilon \tag{177c}$$

where the Sun's longitude, $\nu_{\odot}$, from the mean equinox of date is

$$\nu_{\odot} = 4.8883394 + 0.017202791\ d + 0.03345 \sin\left(6.2482906 + 0.0172019697\ d\right) \tag{178}$$

and the mean obliquity of the ecliptic, ϵ, is obtained from

$$\epsilon = 0.40920619 - 0.6218433 \times 10^{-8}\ d \tag{179}$$

The Sun's distance from the center of Earth can be obtained in kilometers from

$$r_{\odot} = 1.496 \times 10^{8}\left[1 - 0.016725 \cos\left(6.2482906 + 0.0172019697\ d\right)\right] \tag{180}$$

These expressions are based on the fundamental ephemerides as given in Ref. 1. The angles are in radians and d is in ephemeris days from 1950 January 1, 0^h E. T.

E. MOON'S GRAVITATIONAL ATTRACTION

Since the Moon is much closer to the Earth than the Sun, the convergence of the series in the disturbing function will be slower. Therefore, for higher altitude satellites and accurate orbit computations it may be necessary to consider more terms. To a first order accuracy, we can write the accelerations as obtained in the previous section.

$$X_{\zeta} = \frac{\partial R_{\zeta}}{\partial x} = \frac{\mu_{\zeta}}{r_{\zeta}^3} \left[3r\, \xi_{\zeta} \left(\xi\xi_{\zeta} + \eta\eta_{\zeta} + \zeta\zeta_{\zeta} \right) - x \right] \tag{181a}$$

$$Y_{\zeta} = \frac{\partial R_{\zeta}}{\partial y} = \frac{\mu_{\zeta}}{r_{\zeta}^3} \left[3r\, \eta_{\zeta} \left(\xi\xi_{\zeta} + \eta\eta_{\zeta} + \zeta\zeta_{\zeta} \right) - y \right] \tag{181b}$$

$$Z_{\zeta} = \frac{\partial R_{\zeta}}{\partial y} = \frac{\mu_{\zeta}}{r_{\zeta}^3} \left[3r\, \zeta_{\zeta} \left(\xi\xi_{\zeta} + \eta\eta_{\zeta} + \zeta\zeta_{\zeta} \right) - z \right] \tag{181c}$$

Where ξ, η, ζ are defined in the previous section, and ξ_{ζ}, η_{ζ}, ζ_{ζ} are the direction cosines of the Moon's radius vector from the center of the Earth. They can be computed from the following expressions

$$\xi_{\zeta} = \cos \lambda_T \cos \Omega - 0.99596 \sin \lambda_T \sin \Omega \tag{182a}$$

$$\eta_{\zeta} = \cos \epsilon \left(\cos \lambda_T \sin \Omega + 0.99596 \sin \lambda_T \cos \Omega \right) - 0.08976 \sin \epsilon \sin \lambda_T \tag{182b}$$

$$\zeta_{\zeta} = \sin \epsilon \left(\cos \lambda_T \sin \Omega + 0.99596 \sin \lambda_T \cos \Omega \right) + 0.08976 \cos \epsilon \sin \lambda_T \tag{182c}$$

where ϵ is the mean obliquity of the ecliptic given by Eq. (179).

λ_T is the angular position of the Moon measured in the Moon's orbital plane from the ascending node on the ecliptic, and is obtained from the expression

$$\lambda_T = 0.91215551 + 0.23089572\ d + 0.10980098 \sin (3.7617316 + 0.22802714\ d) \quad (183)$$

Ω is the longitude of the ascending node of the lunar orbit on the ecliptic, measured from the mean equinox of date.

$$\Omega = 0.21140807 - 0.0009242193\ d \quad (184)$$

The lunar distance can be computed from

$$r_{\mathbb{C}} = 384400 \left[1 - 0.054900489 \cos (3.7617316 + 0.22802714\ d)\right] \quad (185)$$

In the above equations, the angles are given in radians, $r_{\mathbb{C}}$ is in kilometers, and d is defined in Section VIII-D. The higher order terms have been omitted but the equations are sufficiently accurate for our purposes.

F. SOLAR RADIATION PRESSURE

The energy of solar radiation which is imparted to a celestial body per one square meter of surface area per one second of time is (Ref. 21)

$$S = 1350 \left(\frac{r_0}{r}\right)^2 \frac{\text{joules}}{m^2\ \text{sec}} \quad (186)$$

Where r_0 is the mean distance of Earth from Sun, and r is the distance of the body from Sun.

The force on the body generated by this energy can be expressed as

$$F_{SR} = \frac{S\ (1 + R)}{c} \cos^2 \alpha \quad \left[\frac{\text{newtons}}{m^2}\right] \quad (187)$$

where R is the reflection coefficient, R = 0 for an absolute black body, and R = 1 for a specular surface, c is the velocity of light, and α is the angle of incidence with the surface.

For a sphere, this equation reduces to

$$F_{SR} = \frac{2\ S(1 + R)}{c} \quad \left[\frac{\text{newtons}}{m^2}\right] \quad (188)$$

The acceleration produced by this force will be

$$a_{SR} = F_{SR} \frac{A}{m} \left[\frac{m}{sec^2}\right] \quad (189)$$

where A is the reference area, and m the mass of the body.

The direction of the acceleration is away from the Sun, and thus it opposes the Sun's gravitational attraction. The components of the acceleration are found by multiplying a_{SR} by the direction cosines $\xi_\odot, \eta_\odot, \zeta_\odot$

$$X_{SR} = - a_{SR} \xi_\odot \tag{190a}$$

$$Y_{SR} = - a_{SR} \eta_\odot \tag{190b}$$

$$Z_{SR} = - a_{SR} \zeta_\odot \tag{190c}$$

It must be noted that the direction cosines $\xi_\odot$, $\eta_\odot$, $\zeta_\odot$ are defining the direction Earth-Sun. However, for close satellites, the error introduced by using them for satellite-Sun direction is negligible.

The radiation pressure is experienced by the satellite only at times when it is in the sunlight. The angle θ between the two vectors in the directions to satellite and Sun is found from

$$\cos \theta = \xi\xi_\odot + \eta\eta_\odot + \zeta\zeta_\odot \tag{191}$$

The satellite is always in sunlight

when

$$r \sin\theta > R_E$$

If $r \sin\theta < R_E$, the satellite is in sunlight only when simultaneously $\cos\theta > 0$.

G. INTEGRATION OF THE EQUATIONS OF MOTION

Integration methods can be, generally, placed in three groups: (1) self-starting techniques, (2) difference techniques, and (3) predictor-corrector techniques. Although all three techniques have been used in orbit integration, the self-starting methods have a distinct advantage in orbit determination programs. The advantage stems from the fact that the integration must be done between two arbitrary observation times, but the last two techniques, normally, employ a constant integration interval. Moreover, the differential equations for a close Earth satellite are of second order in which the first derivatives are included. Most of the standard methods of the second and third type used in astronomical orbit integration have been developed for the special case in which the first derivatives are absent (Ref 22). The disadvantage of the self-starting methods lies, mainly, in the computing time required for the integration. However, this can be compensated to a large extent by incorporating methods which vary

the integration step size and/or estimate and adjust the error according to some criterion.

Errors in numerical integration are due to two causes: truncation and round-off. Truncation errors are caused by the replacement of the actual differential equations by ordinary difference equations and omission of higher order terms. The round-off errors are due to the finite number of digits carried in the computations. Not much can be done about the round-off errors besides increasing the number of digits. The truncation errors, however, can be controlled to a certain extent by varying the integration interval. Two types of techiques generally are used for this purpose. One technique estimates the truncation error and adjusts the integration interval and error after an initial integration step has been computed. This involves additional computations and thus, in part, cancels the benefits.

In cases where the nature of the differential equations is known, the criterion can be determined in advance and each step size determined automatically. Thus, considerable savings in time can be gained. A technique of this kind is used in the present application. The rule for the step size is

$$\Delta h = k\, r \tag{192}$$

where Δh is the integration interval, r is the distance from the center of the Earth, and k is a constant for the orbit, determined as follows.

At perigee

$$r\ \Delta\theta = V_{per}\ \Delta h$$

and

$$\Delta h = \frac{\Delta\theta}{V_{per}}\, r = k\, r$$

thus

$$k = \frac{\Delta\theta}{1 + e}\left(\frac{C}{\mu}\right)^{1/2} \tag{193}$$

Where $\Delta\theta$ is dependent on the integration method. It was found that optimum values of $\Delta\theta$ are obtained from the following empirical expression:

$$\Delta\theta = c_1 + c_2 \exp\left(-\, c_3\, e\right) \tag{194}$$

Where e is eccentricity and c_1, c_2, c_3 are constants which depend on the integration method. The criterion gives approximately equal arc lengths for the integration intervals.

The most popular of the self-starting techniques has been the fourth order Runge-Kutta method. A similar method, which compares rather favorably with the Runge-Kutta method, has been developed by Bowie (Ref. 23). This method is based on the assumption that the second derivatives vary quadratically over the integration interval.

In the present effort, these methods are applied in a direct integration mode. In a broader sense it could be classified as the Cowell's method. A different principle is employed in the Encke's method. In this method the coordinates are not obtained directly, but rather the integration is performed on the difference between the actual coordinates and the coordinates of a conic section. The conic section is obtained from the position and velocity components at a particular instant, called the epoch of osculation. The departures from the osculating orbit are called perturbations. If the perturbations are small, they can be expressed by fewer significant figures, which permits larger intervals than with the direct integration methods. When the perturbations increase to an intolerable size, a rectification of the orbit is required. The position and velocity components are determined at a new epoch and the integration restarted.

The application of the Encke's method in close satellite orbit determination programs is impaired because of the irregular intervals between observations which complicates the solution of the two-body equations. In any case, a definite superiority between the Encke and Cowell type methods in modern computer applications has not been established.

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APPENDIX

A. EQUATIONS OF MOTION IN RECTANGULAR COORDINATES

Equations will be developed for motion in an inverse-square central force field. According to Kepler's laws, the path of such motion is a conic section with the central mass occupying one focus. We will first consider an elliptic motion. An ellipse is formed if a point moves in such a manner that the sum 2a of its distances r_1 and r_2 from two fixed points, the focuses, is a constant, i.e.,

$$r_1 + r_2 = 2a = \text{const} \tag{A.1}$$

where a is the semimajor axis. In a Cartesian axes system with the origin at one focus

$$x^2 + y^2 = r_1^2 \equiv r^2 \tag{A.2}$$

and

$$(x - x_F)^2 + (y - y_F)^2 = r_2^2 \tag{A.3}$$

where x_F, y_F are the coordinates of the vacant focus.

Substituting Eqs (A.2) and (A.3) into Eq (A.1) we obtain

$$\frac{x_F}{2a} x + \frac{y_F}{2a} y + \left(a - \frac{x_F^2 + y_F^2}{4a} \right) = (x^2 + y^2)^{1/2} \tag{A.4}$$

In this axes system (Fig. A.1)

$$x_F = 2\epsilon \cos \bar{\omega}$$

$$y_F = 2\epsilon \sin \bar{\omega}$$

where ϵ is the linear eccentricity and $\bar{\omega}$ is the angle between the major axis (positive direction toward apocenter) and x-axis. Hence

$$(2\epsilon)^2 = x_F^2 + y_F^2$$

and evaluating the third term in Eq (A.4) we obtain

$$\frac{a^2 - \epsilon^2}{a} \tag{A.5}$$

Linear eccentricity can be also expressed as

$$\epsilon^2 = a^2 - b^2 \tag{A.6}$$

where b is the semiminor axis. Substituting (A.6) into (A.5) we obtain

$$\frac{b^2}{a} = C \tag{A.7}$$

which is an expression for the semilatus rectum. Equation (A.4) can now be written

$$c \cos \bar{\omega}\, x + e \sin \bar{\omega}\, y + C = r$$

where c is the numerical eccentricity (or simply eccentricity)

$$c = \frac{\epsilon}{a}$$

Substituting $A = e \cos \bar{\omega}$, and $B = e \sin \bar{\omega}$ we obtain

$$A x + B y + C = r \tag{A.8}$$

which is the equation of an ellipse in rectangular coordinates. Differentiating Eq (A.8) with respect to time we obtain

$$(A r - x)\, \dot{x} + (B r - y)\, \dot{y} = 0 \tag{A.9}$$

For a motion in a central force field with the force varying inversely as the square of the distance from the central mass, the following relationship holds

$$h^2 = \mu C$$

where h is the angular momentum about the center of force. Expressed in rectangular coordinates, the angular momentum is

$$h = x \dot{y} - y \dot{x}$$

which gives

$$(x \dot{y} - y \dot{x})^2 = \mu C \tag{A.10}$$

Equations (A.8), (A.9) and (A.10) completely define the motion. Identical equations can be obtained for a parabola and a hyperbola following the same procedure.

From the definition of A and B it follows that

$$e = \left(A^2 + B^2\right)^{1/2} \tag{A.11}$$

For a case $y = 0$, the expressions reduce to

$$(1 - A)\, x_{_} = C \tag{A.12}$$

$$(1 - A)\, \dot{x}_{_} = B\, \dot{y}_{_} \tag{A.13}$$

$$x_{_}\, \dot{y}_{_} = (\mu C)^{1/2} \tag{A.14}$$

Solving for the planar orbital elements A, B, and C

$$A = 1 - \frac{x_\omega \dot{y}_\omega^2}{\mu} \tag{A.15}$$

$$B = \frac{x_\omega \dot{x}_\omega \dot{y}_\omega}{\mu} \tag{A.16}$$

$$C = \frac{(x_\omega \dot{y}_\omega)^2}{\mu} \tag{A.17}$$

Equations (A.8), (A.9) and (A.10) can be solved for the velocity components

$$\dot{x} = \pm \frac{B\,r - y}{r} \left(\frac{\mu}{C}\right)^{1/2} \tag{A.18a}$$

$$\dot{y} = \mp \frac{A\,r - x}{r} \left(\frac{\mu}{C}\right)^{1/2} \tag{A.18b}$$

where the upper signs are for a direct motion, and the lower signs are for a retrograde motion. When $y = 0$, $x \equiv x_\omega = r$ and the equations reduce to

$$\dot{x}_\omega = \pm B \left(\frac{\mu}{C}\right)^{1/2} \tag{A.19}$$

$$\dot{y}_\omega = \mp (A - 1) \left(\frac{\mu}{C}\right)^{1/2} \tag{A.20}$$

B. EQUATION OF TIME

We can write Eq (A.8) in a canonical system in which the $\bar{x}$-axis points toward the pericenter and the coordinates are normalized dividing by C. In this system $A = -e$, $B = 0$, and $C = 1$.

$$-c\,\bar{x} + 1 = \left(\bar{x}^2 + \bar{y}^2\right)^{1/2} \tag{A.21}$$

This equation can be solved for $\bar{y}$

$$\bar{y} = \pm \left[(e^2 - 1)\,\bar{x}^2 - 2e\,\bar{x} + 1\right]^{1/2} \tag{A.22}$$

or for $\bar{x}$

$$\bar{x} = \frac{e - \left[1 + (e^2 - 1)\,\bar{y}^2\right]^{1/2}}{e^2 - 1} \tag{A.23}$$

where only a negative sign before the radical applies in the chosen axes system. The relation between the coordinates $\bar{x}$, $\bar{y}$ and x_ω, y_ω is given by

$$C\,\bar{x} = x_\omega\,\xi_a + y_\omega\,\eta_a \tag{A.24a}$$

$$C\,\bar{y} = -x_\omega\,\eta_a + y_\omega\,\xi_a \tag{A.24b}$$

The normalized area swept out by a radius vector from the $\bar{x}$-axis to a point $P_k\,(\bar{x}_k\,,\,\bar{y}_k)$ is obtained by integration

$$A_k = \int_{\bar{x}=\bar{x}_k}^{\bar{x}=\bar{x}_p} \bar{y}\,d\,\bar{x} + \frac{\bar{x}_k\,\bar{y}_k}{2}$$

and substituting $\bar{y}$

$$A_k = \int_{\bar{x}=\bar{x}_k}^{\bar{x}=\bar{x}_p} \left[(e^2 - 1)\,\bar{x}^2 - 2e\,\bar{x} + 1\right]^{1/2}\,d\bar{x} + \frac{\bar{x}_k\,\bar{y}_k}{2} \tag{A.25}$$

Considering that

$$\bar{x}_p = \frac{1}{1+e}$$

The integral (A.25) can be obtained for three cases

$$A_k = \frac{1}{2(1-e^2)^{3/2}}\left\{\frac{\pi}{2} - \arcsin\left[e + (1-e^2)\,\bar{x}_k\right] - e(1-e^2)^{1/2}\,\bar{y}_k\right\} \quad \text{if } e < 1 \tag{A.26}$$

$$A_k = \frac{1}{2(e^2-1)^{3/2}}\left\{e(e^2-1)^{1/2}\,\bar{y}_k + \ln\left[e - (e^2-1)^{1/2}\,\bar{y}_k - (e^2-1)\,\bar{x}_k\right]\right\}$$

$$\text{if } e > 1 \tag{A.27}$$

and

$$A_k = \frac{\bar{y}_k^3}{3} + \frac{\bar{x}_k\,\bar{y}_k}{2}\,,\quad \text{if } e = 1 \tag{A.28}$$

From Eq (A.10) we can obtain the normalized, constant area rate

$$\Omega = \frac{1}{2}\left(\frac{\mu}{C^3}\right)^{1/2} \tag{A.29}$$

and the solution for time from pericenter to P $(\bar{x}_k, \bar{y}_k)$ is

$$t_k = \frac{A_k}{\Omega} \tag{A.30}$$

Returning to Eq (A.26), it can be shown, utilizing the geometry in Fig. A-2, that

$$\sin \theta_k = e + (1 - e^2)\, \bar{x}_k$$

and

$$\cos \theta_k = (1 - e^2)^{1/2}\, \bar{y}_k$$

Thus θ_k is the complement of the eccentric anomaly E_k

$$E_k = \frac{\pi}{2} - \theta_k \quad \text{if } \bar{y} \geq 0$$

$$E_k = \frac{3}{2}\pi - \theta_k \quad \text{if } \bar{y} < 0$$

and we can write

$$\cos E_k = e + (1 - e^2)\, \bar{x}_k$$

$$\sin E_k = (1 - e^2)^{1/2}\, \bar{y}_k$$

which shows that Eqs (A.26) and (A.30) are another form of Kepler's equation

$$M_k = E_k - e \sin E_k \tag{A.31}$$

where

$$M_k = \left[\mu \left(\frac{1 - e^2}{C}\right)^3\right]^{1/2} \cdot t_k = \frac{t_k}{K_c}$$

Kepler's equation is transcendental in E and can be solved by successive approximations, utilizing the derivative

$$\frac{d E_k}{d t_k} = \frac{1}{K_c(1 - e \cos E_k)} \tag{A.32}$$

Similar considerations apply to the hyperbolic case. Differentiating Eq (A.27) we obtain

$$\frac{d\bar{y}_k}{dt_k} = \frac{B_k \left[(e^2 - 1)\,\bar{x}_k - e\right]}{K_c \left\{(B_k e - 1)\left[(e^2 - 1)\,\bar{x}_k - e\right] - (e^2 - 1)^{1/2}\,\bar{y}_k\right\}} \tag{A.33}$$

where

$$B_k = e - (e^2 - 1)\,\bar{x}_k - (e^2 - 1)^{1/2}\,\bar{y}_k$$

For a parabola, a closed form solution can be obtained. Given time t_k from pericenter to P $(\bar{x}_k, \bar{y}_k)$, the corresponding normalized area is

$$A_k = \frac{t_k}{K_c} \tag{A.34}$$

where

$$K_c = 2\left(\frac{C^3}{\mu}\right)^{1/2}$$

If e = 1, Eq (A.21) gives

$$\bar{x}_k = \frac{1 - \bar{y}_k^2}{2} \tag{A.35}$$

Substituting Eq (A.35) into (A.28) we obtain a cubic in $\bar{y}_k$

$$\bar{y}_k^3 + 3\,\bar{y}_k - 12A_k = 0$$

The discriminant of this equation, $\Delta > 0$, and, therefore, it will always have one real root

$$\bar{y}_k = \left[6A_k + (36A_k^2 + 1)^{1/2}\right]^{1/3} + \left[6A_k - (36A_k^2 + 1)^{1/2}\right]^{1/3} \tag{A.36}$$

and $\bar{x}_k$ is obtained from Eq (A.35).

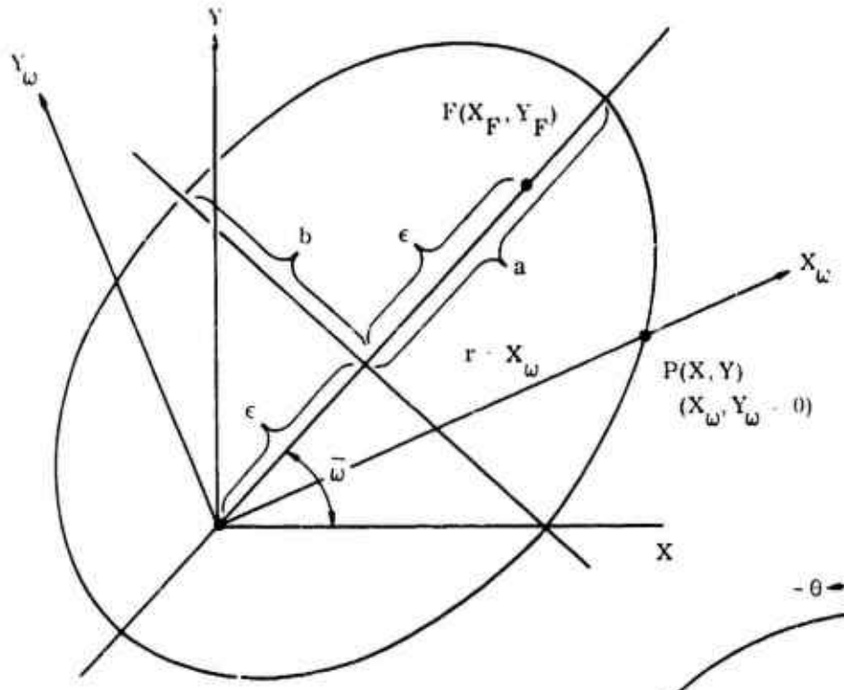

Fig. A-1. Axes Systems

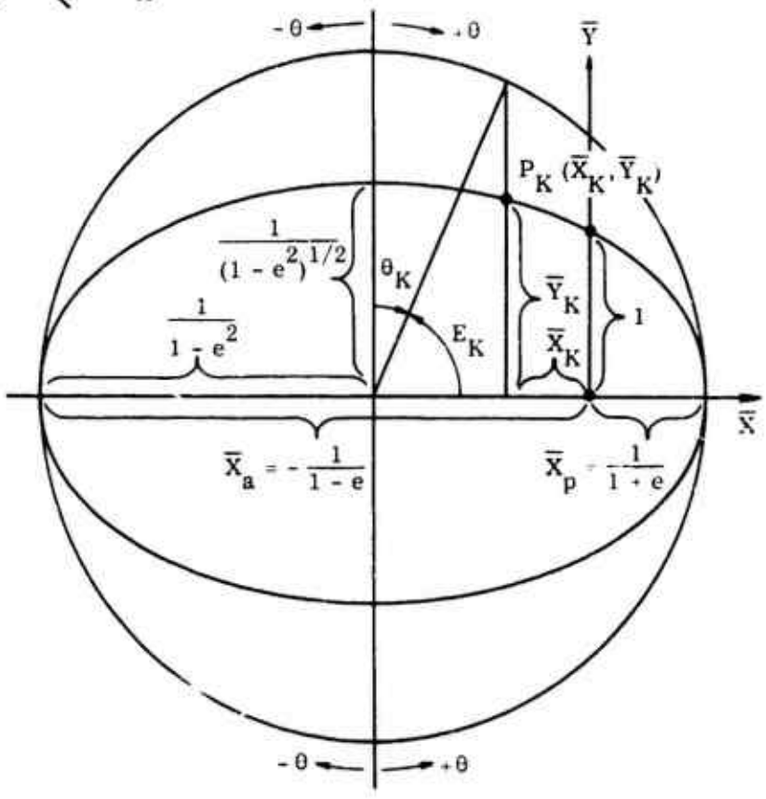

Fig. A-2. Ellipse in a Canonical System

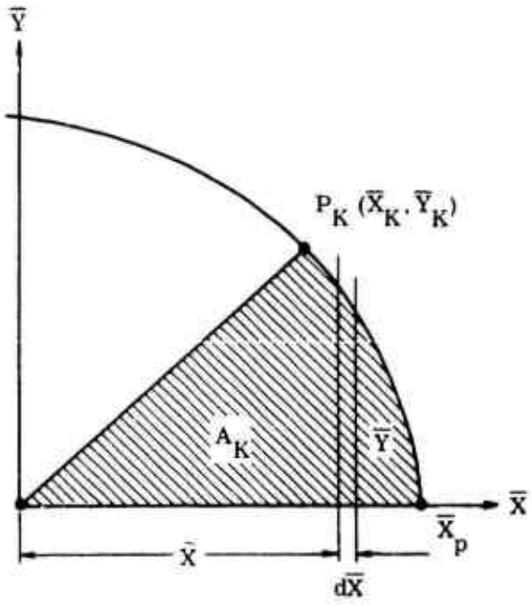

Fig. A-3. Normalized Area

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13. ABSTRACT

The aspects of accurate determination of Earth satellite orbits by the Minimum Variance Method are presented. In addition, techniques for the determination of the associated physical constants, such as the coefficients in the Earth's gravitational potential, exospheric temperature, etc., are developed. A method for determination of the state transition matrix is presented. Also included are a review of the time systems employed in satellite orbit determination and a short discussion of the types of observations.

The mathematical model of the dynamical system includes nine zonal harmonics and up to the fourth order tesseral harmonics of the Earth's gravitational potential. Atmospheric drag effects are included on the assumption that the atmosphere rotates with the angular velocity of the Earth. First order solar and lunar gravitational attractions and solar radiation pressure are also treated. The satellite orbits are integrated in a reference system which considers the precession and nutation of the Earth. Rectangular coordinate systems are used throughout the development.

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